Estimation of stable spectral measures

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Abstract

We present two new estimators of a stable spectral measure. One is based on the empirical characteristic function, the other is based on one dimensional projections of the data. We compare these estimators with the Rachev-Xin-Cheng estimator in an empirical study. Their applications in modeling financial portfolios are also discussed.

Keywords: Multivariate α-stable distributions, spectral measure, estimation, financial modeling.

1 Introduction

Recently there has been increasing interest in the use of multivariate α-stable ($0 < \alpha < 2$) distributions in a variety of practical problems. Our interest in

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α-stable vectors is motivated by their applications to modeling financial portfolios. Since the seminal works of Mandelbrot (1963) and Fama (1965), stable models for financial asset returns have occupied a prominent place in statistical and financial literature (see Akgiray and Booth (1988), Kozubowski and Rachev (1994), Mittnik and Rachev (1991) and (1993), Panorska (1996)). Although the estimation problem for the parameters of the univariate stable law seems to have been satisfactorily solved (see DuMouchel (1971), McCulloch (1986), Nolan (1997b)), not much is known at present about the statistics of multivariate stable distributions. The main difficulty in dealing with stable laws, both univariate and multivariate, is that except for a few special cases, they do not have densities that can be written in a closed form. The only available information for stable random vectors is in the form of their characteristic functions. This is a major problem in working with the stable laws, because it causes difficulties or failure of many traditional methods of estimation, simulation, etc. However, some multivariate problems have been solved. An estimation of parameters procedure and definition of risk of a stable portfolio was given in Press (1972a, b). A method of simulating random samples from multivariate stable distributions was presented by Modarres and Nolan (1994), and a method for approximation of a stable spectral measure by a discrete measure and numerical calculation of multivariate stable densities was given in Byczkowski, Nolan, and Rajput (1991) and Nolan and Rajput (1995).

This work was inspired by Rachev and Xin (1993) and Cheng and Rachev (1994), who investigated estimation of the spectral measure and association problems for samples from the domain of attraction of a multivariate stable law. The need for an efficient and numerically sound estimator of a multivariate α-stable spectral measure arises in connection with stochastic modeling of financial portfolios. The spectral measure carries essential information about the vector, in particular about the dependence structure between the individual stocks that make up the portfolio.

We begin with preliminary definitions and notation. We treat a portfolio of assets as a multivariate α-stable ($0 < \alpha < 2$) vector $\mathbf{X} \in \mathbf{R^d}$ with characteristic function (ch.f.)

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbf{E} \exp\{i < \mathbf{X}, \mathbf{t} >\} = \exp(-I_{\mathbf{X}}(\mathbf{t})),$$

where the exponent function is given by

$$I_{\mathbf{X}}(\mathbf{t}) = \int_{S^d} \psi_\alpha(< \mathbf{t}, \mathbf{s} >) \Gamma(ds) + i < \mu, \mathbf{t} > .$$

Here S^d is the unit sphere in R^d, Γ is the spectral measure of the vector $\mathbf{X}$, μ is a vector in R^d, $< \mathbf{t}, \mathbf{s} >= t_1 s_1 + \cdots + t_d s_d$ is the inner product, and

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha (1 - i \ \mathrm{sign}(u) \tan \frac{\pi\alpha}{2}) & \alpha \neq 1 \\ |u|(1 + i \frac{2}{\pi} \ \mathrm{sign}(u) \ln |u|) & \alpha = 1. \end{cases}$$

We note that this standard parameterization is discontinous at $\alpha = 1$ (unless one restricts to the symmetric case), since $|\tan \frac{\pi\alpha}{2}| \to \infty$ as $\alpha \to 1$. A consequence of this is that estimates of μ and Γ are likely to be poor whenever α is near 1. This problem can be avoided by using the multivariate version of Zolotarev's (M) parameterization, see Zolotarev (1986) or Nolan (1996).

Our objective is to estimate the spectral measure Γ and the shift vector μ, given an iid sample $\mathbf{X}_1, \ldots, \mathbf{X}_k$ of d-dimensional r. vectors drawn from this distribution. We will first shift to eliminate μ, then the rest of the paper will focus on estimating the spectral measure. Note that in financial applications, $1 < \alpha \leq 2$, and the mean of the data vectors is a simple estimator of μ. For a general α, we assume the availability of some method of estimating parameters of a one dimensional data set of stable random variables. The best method for doing this is to use maximum likelihood estimation (MLE). While several researchers have used this approach, it is prohibitively expensive (in computational time) at present. The first attempt at MLE of stable parameters is in the work of DuMouchel (1971). Efficient programs now exist for approximating symmetric ($\beta = 0$) stable densities for $\alpha > 0.85$ and it is possible to estimate stable parameters by MLE in the symmetric case, see McCulloch (1997). There are now reliable programs for computing general stable densities, see Nolan (1997a), and a program to compute MLE of all four stable parameters, see Nolan (1997b). Since the methods below require many one dimensional estimates of stable parameters, we needed a fast method for estimating parameters of general stable distributions. For this reason we used the quantile based estimators of McCulloch (1986) in the examples below.

Using some method, estimate the one-dimensional parameters $(\hat{\alpha}_j, \hat{\sigma}_j, \hat{\beta}_j, \hat{\mu_j})$, $j = 1, \ldots, d$ for each of the coordinates of the d-dimensional data set. The vector $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_d)$ is an estimate of the shift vector. Furthermore, define $\bar{\alpha} = \left(\sum_{j=1}^d \hat{\alpha}_j\right)/d$ as an estimate of the joint index of stability α. For the rest of this paper, we assume that the data has been shifted so that μ is zero, and that we know α.

Throughout this paper, we assume that the underlying distribution is multivariate stable. Diagnostics for assessing the joint stability of a sample

are discussed in Nolan and Panorska (1996). If the data are not close to stable, then the methods described below should not be used. One departure from joint stability is marginal stability, i. e. when the components of $\mathbf{X}$ (individual stocks) have different indices of stability, see Cambanis and Taraporevala (1995). A more serious departure is when there are gaps in the support of the distribution, multimodality, etc. yet still heavy tails. In this situation, the distribution may be in the domain of attraction of a stable distribution and Rachev-Xin-Cheng (RXC) method is more robust. See the Section 4 for a discussion of issues related to this.

The paper is organized as follows. In Section 2, we present two new estimators of the spectral measure. Section 3 is devoted to the empirical comparison of the performance of the estimators on simulated random samples from known spectral measures, and to the estimation of the spectral measure from a bivariate sample of foreign exchange rates. We conclude the paper with a discussion of the results in Section 4.

2 Estimation of the spectral measure

2.1 Empirical characteristic function method and projection method.

The two new estimators are based on using the sample to estimate the ch. f. on some grid. More precisely, we will estimate the exponent of the ch. f. $I_{\mathbf{X}}(\cdot)$ on a grid $\mathbf{t}_1, \ldots, \mathbf{t}_n \in S^d$. Below we show how to recover an estimate of the spectral measure from this.

The empirical characteristic function (ECF) method is straightforward. Given an i.i.d. sample $\mathbf{X}_1, \ldots, \mathbf{X}_k$ of α-stable random vectors with spectral measure Γ, let $\widehat{\phi}_k(\mathbf{t})$ and $\widehat{I}_k$ be the empirical counterparts of ϕ and I, i.e. $\widehat{\phi}_k(\mathbf{t}) = (1/k)\sum_{i=1}^{k} \exp(i < \mathbf{t}, \mathbf{X}_i >)$ is the sample characteristic function, and $\widehat{I}_k(\mathbf{t}) = -\ln \widehat{\phi}_k(\mathbf{t})$. Given a grid $\mathbf{t}_1, \ldots, \mathbf{t}_n \in S^d$, $\vec{I}_{ECF,k} = \left[\hat{I}_k(\mathbf{t}_1), \ldots, \hat{I}_k(\mathbf{t}_n)\right]'$ is the ECF estimate of $I_{\mathbf{X}}(\cdot)$. $\alpha_{ECF} = \bar{\alpha}$ defined above (average of coordinate α's) is used as the estimate of α in the ECF method.

The second method was defined by McCulloch (1994); we call it a projection method, because it is based on one-dimensional projections of the data set. First consider a projection of the r. vector $\mathbf{X}$: for any $\mathbf{t} \in S^d$, $< \mathbf{t}, \mathbf{X} >$ is a one-dimensional r. variable with characteristic function $E \exp(iu < \mathbf{t}, \mathbf{X} >) = \exp(-I_{\mathbf{X}}(u\mathbf{t}))$. Hence it's scale, skewness and shift are given by (pg. 20

of Zolotarev (1986) or Example 2.3.4 of Samorodnitsky and Taqqu (1994)):

$$\begin{aligned}
\sigma^\alpha(\mathbf{t}) &= \Re I_{\mathbf{X}}(\mathbf{t}) = \int_{S^d} |<\mathbf{t},\mathbf{s}>|^\alpha \Gamma(d\mathbf{s}), \\
\beta(\mathbf{t}) &= \sigma^{-\alpha}(\mathbf{t}) \int_{S^d} \operatorname{sign}<\mathbf{t},\mathbf{s}> |<\mathbf{t},\mathbf{s}>|^\alpha \Gamma(ds) \\
&= \begin{cases} -\Im I_{\mathbf{X}}(\mathbf{t})/(\sigma^\alpha(\mathbf{t}) \tan\frac{\pi\alpha}{2}) & \alpha \neq 1 \\ \Im[I_{\mathbf{X}}(2\mathbf{t}) - 2I_{\mathbf{X}}(\mathbf{t})]/(4\sigma(\mathbf{t}) \ln 2/\pi) & \alpha = 1 \end{cases} \\
\mu(\mathbf{t}) &= \begin{cases} 0 & \alpha \neq 1 \\ -\frac{2}{\pi} \int <\mathbf{t},\mathbf{s}> \ln|<\mathbf{t},\mathbf{s}>| \Gamma(ds) = -\Im I_{\mathbf{X}}(\mathbf{t})/\sigma(\mathbf{t}) & \alpha = 1 \end{cases}
\end{aligned}$$

Now consider the sample $\mathbf{X}_1, \ldots, \mathbf{X}_k$. Fix a grid $\mathbf{t}_1, \ldots, \mathbf{t}_n$ on S^d, and for each $\mathbf{t}_j$, define the one-dimensional data set $<\mathbf{t}_j, \mathbf{X}_1>, \ldots, <\mathbf{t}_j, \mathbf{X}_k>$. Use some method to estimate the scale $\hat{\sigma}(\mathbf{t}_j)$ and skewness $\hat{\beta}(\mathbf{t}_j)$ (and shift $\hat{\mu}(\mathbf{t}_j)$ when $\alpha = 1$) of this one-dimensional data. Define

$$\hat{I}_k(\mathbf{t}_j) = \begin{cases} \hat{\sigma}^\alpha(\mathbf{t}_j) \left(1 - i\hat{\beta}(\mathbf{t}_j) \tan\frac{\pi\alpha}{2}\right) & \alpha \neq 1 \\ \hat{\sigma}(\mathbf{t}_j) \left(1 - i\hat{\mu}(\mathbf{t}_j)\right) & \alpha = 1 \end{cases}$$

The vector $\vec{I}_{PROJ,k} = \left[\hat{I}_k(\mathbf{t}_1), \ldots, \hat{I}_k(\mathbf{t}_n))\right]'$ is the projection estimator of $I_{\mathbf{X}}(\cdot)$. Since we estimated the parameters of each projection, we also get an estimate $\hat{\alpha}(\mathbf{t}_j)$ for each direction. In the examples below, we used the average $\alpha_{PROJ} = (1/n) \sum \hat{\alpha}(\mathbf{t}_j)$ as a pooled estimate of α for the PROJ method.

2.2 Recovering the spectral measure

To simplify notation, in the remainder of this paper we consider the sample size k fixed and supress the dependence of the estimators on the sample size. That is, we write $\vec{I}_{ECF}$ for $\vec{I}_{ECF,k}$, etc. In order to obtain an estimate of the spectral measure $\widehat{\Gamma}$, we need to invert the discrete approximations to the characteristic function obtained by the ECF or $PROJ$ methods.

We start with the case when Γ is a discrete spectral measure, i.e.,

$$\Gamma(\cdot) = \sum_{j=1}^{n} \gamma_j \delta_{\mathbf{s}_j}(\cdot), \tag{1}$$

where γ_j's are the weights, and $\delta_{\mathbf{s}_j}$'s are point masses at the points $\mathbf{s}_j \in S^d, j = 1, \ldots, n$. In this case, $I_{\mathbf{X}}(\mathbf{t}) = \sum_{j=1}^{n} \psi_\alpha(<\mathbf{t}, \mathbf{s}_j>)\gamma_j$. Further, let

$\mathbf{t}_1, \ldots, \mathbf{t}_n \in R^d$, and define the $n \times n$ matrix

$$\Psi = \Psi(\mathbf{t}_1, \ldots, \mathbf{t}_n; \mathbf{s}_1, \ldots, \mathbf{s}_n) = \begin{bmatrix} \psi_\alpha(< \mathbf{t_1}, \mathbf{s_1} >) & , \ldots , & \psi_\alpha(< \mathbf{t_1}, \mathbf{s_n} >) \\ \cdot & \cdot & \cdot \\ \psi_\alpha(< \mathbf{t_n}, \mathbf{s_1} >) & , \ldots , & \psi_\alpha(< \mathbf{t_n}, \mathbf{s_n} >) \end{bmatrix}.$$

If $\vec{\gamma} = [\gamma_1, \ldots, \gamma_n]'$, and $\vec{I} = [I_{\mathbf{X}}(\mathbf{t}_1), \ldots, I_{\mathbf{X}}(\mathbf{t}_n)]'$, then

$$\vec{I} = \Psi \vec{\gamma}. \tag{2}$$

If $\mathbf{t_1}, \ldots, \mathbf{t_n} \in R^d$ are chosen so that Ψ^{-1} exists, then $\vec{\gamma} = \Psi^{-1}\vec{I}$ is the exact solution of (2).

For a general spectral measure Γ (not discrete and/or the location of the point masses are unknown), consider a discrete approximation $\Gamma^* = \sum_{j=1}^n \gamma_j \delta_{\mathbf{s_j}}$, where $\gamma_j = \Gamma(A_j)$, $j = 1, \ldots, n$ are the weights, and $\delta_{\mathbf{s}_j}$'s are point masses. When $d = 2$, it is natural to take $\mathbf{s}_j = (\cos(2\pi(j-1)/n), \sin(2\pi(j-1)/n)) \in S^d$, and arcs $A_j = (2\pi(j-(3/2))/n, 2\pi(j-(1/2))/n]$, $j = 1, \ldots, n$. In higher dimensions, the A_j's are patches that partition the sphere S^d, with some "center" $\mathbf{s}_j$. In this case, each of the coordinates of the $\vec{\gamma} = [\gamma_1, \ldots, \gamma_n]'$ is an approximation to the mass of the patch containing $\mathbf{s}_j$, $j = 1, \ldots, n$.

The principle behind the estimation of Γ is simple: given some grid $\mathbf{t}_j = \mathbf{s}_j$, $j = 1, \ldots, n$ and either estimate ($\vec{I}_{ECF}$ or $\vec{I}_{PROJ}$) of $\vec{I}$, invert (2) to get $\vec{\gamma}$. Using these weights and the grid $\mathbf{s}_1, \ldots, \mathbf{s}_n$, define $\hat{\Gamma}$ by (1).

While the above method is formally correct, it has serious drawbacks. The first is that the solutions we get for $\vec{\gamma}$ will generally be complex numbers, since the matrix Ψ is complex and the vectors $\vec{I}_{ECF}$ and $\vec{I}_{PROJ}$ will have some noise in them. The second is that if the grid $\mathbf{t}_1, \ldots, \mathbf{t}_n$ has an even number of points and they are spread uniformly around S^d, then $\mathbf{t}_j = -\mathbf{t}_m$ for some j, m, in which case the linear system is singular (because $\psi(-u) = \overline{\psi(u)}$, and $I_{\mathbf{X}}(-\mathbf{t}) = \overline{I_{\mathbf{X}}(\mathbf{t})}$). This is a troublesome drawback, because we generally want n to be a multiple of 2^d; e.g. when $d = 2$, we would like to estimate the mass around each point (1,0), (0,1), (-1,0), and (0,-1), which correspond to independent components.

In fact, we can use the properties of ψ and $I_{\mathbf{X}}$ and a "symmetric" grid to our advantage by restating the problem without complex numbers. This conveniently eliminates the problem of imaginary weights γ_j. We state the method in R^2. Assume $n = 2m$ and the grid is given by $\mathbf{t}_j = \mathbf{s}_j = (\cos(2\pi(j-1)/n), \sin(2\pi(j-1)/n))$. Then the components of $\vec{I}$ satisfy

$I_j = \overline{I}_{j+m}$ and the entries of Ψ satisfy $\psi_{i,j} = \psi_\alpha(< \mathbf{t}_i, \mathbf{t}_j >) = \psi_\alpha(\cos(2(i-j)\pi/n)) = \overline{\psi}_{i,j+m}$. Then for $j = 1, \ldots, m$, $\Re I_j = (I_j + I_{j+m})/2 = (\sum_{k=1}^n \psi_{j,k}\gamma_k + \sum_{k=1}^n \psi_{j+m,k}\gamma_k)/2 = \sum_{k=1}^n ((\psi_{j,k}\psi_{j+k,k})/2)\gamma_k = \sum_{k=1}^n \Re\psi_{j,k}\gamma_k$. Likewise, for $j = 1, \ldots, m$, $\Im I_j = -(I_j - I_{j+m})/2 = \sum_{k=1}^n \Im\psi_{j,k}\gamma_k$. So if we define the (real) vector $\vec{c} = (\Re I_1, \ldots, \Re I_m, \Im I_1, \ldots, \Im I_m)'$ and the (real) $n \times n$ matrix $A = [a_{i,j}]$ by

$$a_{i,j} = \begin{cases} \Re\psi_{i,j} & i = 1, \ldots, m \\ \Im\psi_{i,j} & i = m+1, \ldots, n, \end{cases}$$

then

$$\vec{c} = A\vec{\gamma}. \tag{3}$$

The matrix A is nonsingular, so the system is invertible. The linear solutions to the estimation problem are $A^{-1}\vec{I}_{ECF}$, and $A^{-1}\vec{I}_{PROJ}$.

Unfortunately, this restatement of the problem does not eliminate all difficulties. Because of noise in the sample and the fact that A is ill-conditioned, when the real system (3) is inverted, we may get negative values for some of the weights. In practice, we routinely observed alternating signs in the weights, resulting in unreliable estimates of the spectral measure. To avoid this problem, McCulloch (1994) suggested restating the system as a constrained quadratic programming problem that guarantees nonnegative weights:

$$\text{minimize} \qquad \|\vec{c} - A\vec{\gamma}\|^2 = (\vec{c} - A\vec{\gamma})(\vec{c} - A\vec{\gamma})' \quad \text{subject to} \quad \vec{\gamma} \geq 0.$$

We call the solutions obtained by this method $\vec{\gamma}_{ECF}$ when $\vec{I}_{ECF}$ is used to compute $\vec{c}$ in the above equation and $\vec{\gamma}_{PROJ}$ when $\vec{I}_{PROJ}$ is used.

3 Examples

We shall consider four examples of estimating the spectral measure for bivariate data. In the first three the data is simulated using the Modarres and Nolan (1994) procedure, the fourth one estimates the spectral measure for a real sample consisting of foreign exchange rates. The two procedures discussed in the previous sections and the procedure of Rachev-Xin-Cheng (RXC) were used to recover spectral measures. In all cases the linear solutions oscillated wildly, and we did not include them on the graphs. In all cases, we used 100 evenly spaced gridpoints $\mathbf{t}_j, j = 1, \ldots, 100$ where we estimated the weights γ_j. All four examples took less than a minute of execution time on a PC (60 MHz Pentium). Simulated samples had 10,000 data vectors. We turn to the description of examples now.

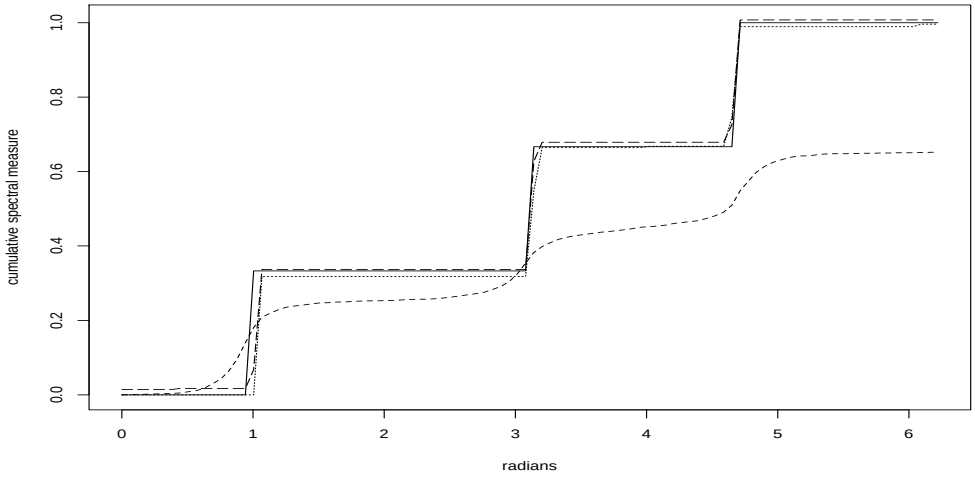

Figure 1: Estimates of the cumulative spectral measure for simulated data from Γ_1. Solid line is the exact cumulative, short dashed line is the Rachev-Xin-Cheng estimator, dotted line is the projection estimator, long dashed line is the characteristic function estimator. (10,000 data vectors)

Example 1. Data was simulated from the measure $\Gamma_1(\cdot) = \sum_{j=1}^{3}(1/3)\delta_{\mathbf{s}_j}(\cdot)$, where $\mathbf{s}_j = (\cos\theta_j, \sin\theta_j) \in S^d, j = 1, \ldots, 3$, and $\theta_1 = \pi/3, \theta_2 = \pi$, and $\theta_3 = 3\pi/2$. The shape parameter is $\alpha = 1.25$. The estimated shape parameters for the three methods were: $\alpha_{RXC} = 1.383$, $\alpha_{PROJ} = 1.275$, and $\alpha_{ECF} = 1.270$. The results of estimation of the cumulative spectral measure are shown in Figure 1. Both the ECF and PROJ methods do a good job of recovering Γ - both the location and the size of the point masses are accurately determined. The RXC method seriously underestimates the total mass, smooths out the mass instead of identifying point masses and overestimates α. The same comments hold for Examples 2 and 3, and presumably hold for the real data in Example 4.

Example 2. Data was simulated from the measure corresponding to independent components with $\alpha = 0.75$: $\Gamma_2(\cdot) = \sum_{j=1}^{4}(1/4)\delta_{\mathbf{s}_j}(\cdot)$, where $\mathbf{s}_j = (\cos((j-1)\pi/2), \sin((j-1)\pi/2))) \in S^d, j = 1, \ldots, 4$. The estimated shape

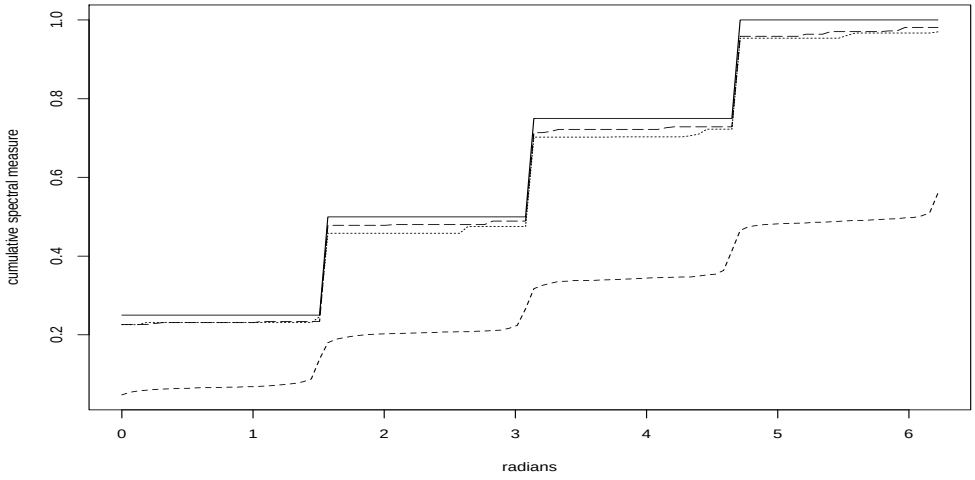

Figure 2: Estimates of the cumulative spectral measure for data simulated from Γ_2. Line types are as in Figure 1. (10,000 data vectors)

parameters for the three methods were: $\alpha_{RXC} = 0.649$, $\alpha_{PROJ} = 0.748$, and $\alpha_{ECF} = 0.755$. The estimated spectral measures are shown in Figure 2.

Example 3. The true measure is smoother then in the first two examples: $\Gamma_3(\cdot) = \sum_{j=0}^{88} \gamma_j \delta_{\mathbf{s}_j}(\cdot)$, where $\mathbf{s}_j = (\cos(2\pi j/90), \sin(2\pi j/90)) \in S^d$, and $\gamma_j = |\sin(4\pi j/90)|$, for $j = 0, \ldots, 88$. The shape parameter is $\alpha = 1.5$. The estimated shape parameters for the three methods were: $\alpha_{RXC} = 1.612$, $\alpha_{PROJ} = 1.499$, and $\alpha_{ECF} = 1.498$. The results of estimation of the cumulative spectral measure are similar to the results in Example 1, and are shown in Figure 3.

Example 4. We looked at a portfolio of two assets: German Mark vs US Dollar (DMUS), and Japanese Yen vs US Dollar (JYUS) exchange rates. The data is in the form of daily log-returns, and spans the period from January 1, 1980 to December 7, 1990 (total of 2563 values). We plotted the results of estimation for this data in Figure 4. The estimated shape parameters for

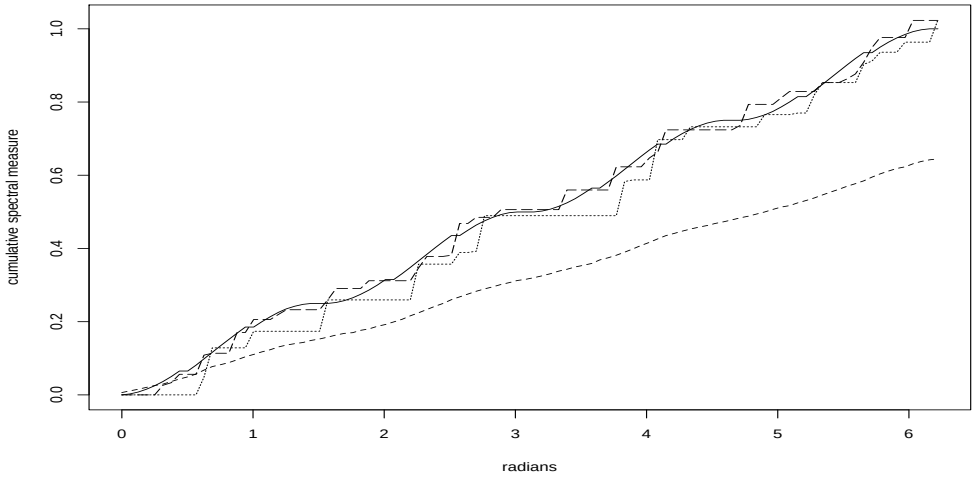

Figure 3: Estimates of the cumulative spectral measure for data simulated from Γ_3. Line types are as in Figure 1. (10,000 data vectors)

the three methods were: $\alpha_{RXC} = 1.579$, $\alpha_{PROJ} = 1.531$, and $\alpha_{ECF} = 1.529$.

Experimentation with other simulated data suggests that the error in the methods comes from 2 sources: random sample fluctuations and discretizing the measure. The first source is not under our control, the second source is. There is a tension between wanting to get fine information about the spectral measure versus having to estimate too many parameters. Experimenting with discrete spectral measures that don't fit the chosen grid showed that the new procedures shifted weights slightly, but were relatively robust in placing estimated masses. Having a fine grid increases computational time, but does not seem to introduce ill-conditioning (at least when the quadratic programming approach is used). These issues are explored further in Nolan and Panorska (1996).

We also observed a problem with the empirical characteristic function method. When the length of the data vectors are very large or very small, the empirical characteristic function gives poor performance. This appears to be a numerical problem with the terms $\exp(i < \mathbf{X}, \mathbf{t} >) = \cos < \mathbf{X}, \mathbf{t} >$

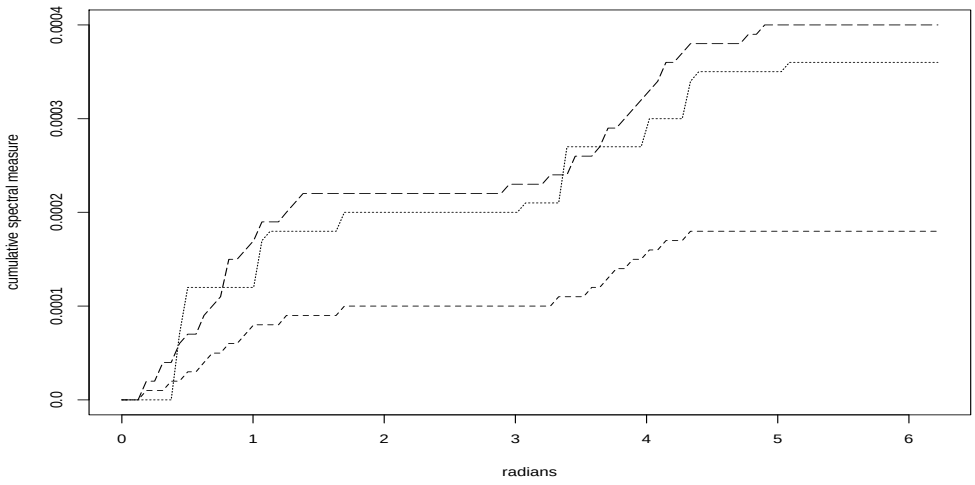

Figure 4: Estimates of the cumulative spectral measure corresponding to the exchange rates data. Line types are as in Figure 1. (2,563 data vectors)

$+i \sin < \mathbf{X}, \mathbf{t} >$. When $|\mathbf{X}|$ is very large the terms oscillate very fast and when $|\mathbf{X}|$ is very small, they hardly oscillate at all, so fixed precision arithmetic will not capture the behavior of $I_{\mathbf{X}}(\mathbf{t})$. We found empirically that scaling the data by the median of the values $|\mathbf{X}_1|, \ldots, |\mathbf{X}_k|$ gave consistently good results across all scales.

We note that all the methods were implemented in double precision arithmetic, which gives approximately 15 decimal digits of significance. This accuracy is especially useful for the ECF method where the sample characteristic function is delicate.

4 Discussion

The empirical characteristic function and the projection method both performed well on simulated data. We recommend always using the quadratic programming solution to recover Γ, and scaling the ECF method by the median value mentioned above.

In the simulated examples we tried (including others not shown here), the Rachev-Xin-Cheng estimator always underestimated the total mass of

the spectral measure and sometimes poorly estimated α. (For large α RXC regularly overestimated α, for small α RXC regularly underestimated α.) If the goal is to identify positive/negative association, then the RXC method works well and is more robust to departures from joint stability. If the goal is sharp estimates of α and/or the total mass, then the RXC method needs to be adjusted. The RXC method is based on tails of the sample: distances of each sample point from the origin (after centering) are computed and the tail is all sample points beyond a certain distance. Those authors use the most extreme k_1 tail values to estimate α and the most extreme k_2 tail values to estimate the total mass of Γ. Our implementation used the suggested value of 20% of the sample for both k_1 and k_2. It is likely that one can improve the performance of the RXC method by adjusting the values of k_1 and k_2. Unfortunately, it is not clear what values to use for k_1 and k_2 - they essentially depend on when the tail density of the multivariate distribution is close to Pareto. Fofack and Nolan (1997) show that in the one dimensional stable case this value is a complex function of α and the skewness. For example, as α approaches 2, only the extreme tail, less than 0.1% of the distribution, has the limiting Pareto behavior. We know of no results on this issue in the multivariate stable case. For a distribution in the domain of attraction of a stable, it is impossible to make any general statement about when the Pareto tail behavior occurs without making assumptions on the data.

The RXC estimate of Γ does place spectral mass in roughly the right locations, but it will always smooth the mass out and consequently not identify point masses. This will not make much difference if the true spectral measure is smooth, but it is important for identifying the case of independent components (e.g. Example 2). We do not see a satisfactory solution to this problem. The RXC method is based on counting how many large (in length) data vectors fall in cones determined by a grid. Making the grid finer will not eliminate the problem since the spread of mass is determined by the spatial distribution of the data. Taking a smaller fraction of the extreme data vectors (20% was used in the examples above) would presumably help this issue, but then the estimate is based on less and less of the data. Reducing the fraction used by too much seems unproductive, unless a very large data set is available.

For comparison, note that the ECF method uses all the data. The PROJ method formally uses all the data, but the fractile method we use to estimate parameters of the one-dimensional projections is based only on the 0.05, 0.25, 0.50, 0.75 and 0.95 fractiles of the projected data. (A MLE implementation

of the PROJ method would use all the data.)

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