An overview of multivariate stable distributions

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1 Introduction

A d-dimensional α-stable random vector is determined by a spectral measure Γ (a finite Borel measure on S_d=unit sphere in $\mathbb{R}^d$) and a shift vector $\mu^0 \in \mathbb{R}^d$, e.g. Samorodnitsky and Taqqu (1994). The notation $\mathbf{X} \sim S_{\alpha,d}(\Gamma, \mu^0)$ will be used to denote such a stable random vector. Until recently, there has been little understanding of what multivariate stable distributions look like, nor any methods for working with them. In this paper we review recent work that makes it possible to use multivariate stable models for practical problems. Our goal here is make these results available to people who want to understand and use multivariate stable densities, so we state results without proof; references are given for those who are interested.

Below we summarize some basic facts about multivariate stable distributions. In Section 2 we describe the class of stable random variables that correspond to discrete spectral measures with a finite number of point masses and show that they are dense in the set of all stable measures. Then in Section 3, formulas are given for computing multivariate stable densities and examples of such computations are given. A method of generating stable random vectors is given in Section 4. Section 5 is concerned with the problem of estimating a spectral measure from a data set, and the last section discusses the problem of identifying the type of a stable process by looking at the spectral measure associated with finite dimensional distributions of the process.

The characteristic function of $\mathbf{X} \sim S_{\alpha,d}(\Gamma, \mu^0)$ is

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbf{E} \exp\{i < \mathbf{X}, \mathbf{t} >\} = \exp(-I_{\mathbf{X}}(\mathbf{t}) + i < \mu^0, t >), \tag{1}$$

where the function in the exponent is

$$I_{\mathbf{X}}(\mathbf{t}) = \int_{S_d} \psi_\alpha(< \mathbf{t}, \mathbf{s} >) \Gamma(ds). \tag{2}$$

Here $< \mathbf{t}, \mathbf{s} >= t_1 s_1 + \cdots + t_d s_d$ is the inner product, and

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha (1 - i \text{sign } (u) \tan \frac{\pi\alpha}{2}) & \alpha \neq 1 \\ |u|(1 + i \frac{2}{\pi} \text{sign } (u) \ln |u|) & \alpha = 1. \end{cases}$$

Since the ch. f. is determined by $I_{\mathbf{X}}(t)$, $t \in \mathbb{R}^d$, the (complex valued) function $I_{\mathbf{X}}(t)$ determines the distribution of $\mathbf{X}$. This idea is used below to estimate Γ from data.

For any $\mathbf{t} \in \mathbb{R}^d$, the projection of the r. vector $< \mathbf{t}, \mathbf{X} >$ is a one-dimensional r. variable with characteristic function $E \exp(iu < \mathbf{t}, \mathbf{X} >) = \exp(-I_{\mathbf{X}}(u\mathbf{t}))$. Hence it's scale, skewness and shift are given by (Zolotarev (1986), pg. 20, or Samorodnitsky and Taqqu (1994), Example 2.3.4):

$$\begin{aligned} \sigma^\alpha(\mathbf{t}) &= \Re I_{\mathbf{X}}(\mathbf{t}) = \int_{S_d} | < \mathbf{t}, \mathbf{s} > |^\alpha \Gamma(d\mathbf{s}), \\ \beta(\mathbf{t}) &= \sigma^{-\alpha}(\mathbf{t}) \int_{S_d} \text{sign} < \mathbf{t}, \mathbf{s} > | < \mathbf{t}, \mathbf{s} > |^\alpha \Gamma(d\mathbf{s}) \end{aligned} \tag{3}$$

$$= \begin{cases} -\Im I_{\mathbf{X}}(\mathbf{t})/(\sigma^\alpha(\mathbf{t})\tan\frac{\pi\alpha}{2}) & \alpha \neq 1 \\ \Im[I_{\mathbf{X}}(2\mathbf{t}) - 2I_{\mathbf{X}}(\mathbf{t})]/(4\sigma(\mathbf{t})\ln 2/\pi) & \alpha = 1 \end{cases}$$

$$\mu(\mathbf{t}) = \begin{cases} <\mathbf{t}, \mu^0> & \alpha \neq 1 \\ <\mathbf{t}, \mu^0> -\frac{2}{\pi}\int <\mathbf{t},\mathbf{s}> \ln|<\mathbf{t},\mathbf{s}>|\,\Gamma(d\mathbf{s}) = -\Im I_{\mathbf{X}}(\mathbf{t})/\sigma(\mathbf{t}) & \alpha = 1 \end{cases}$$

By replacing $\mathbf{X}$ with $\mathbf{X} - \mu^0$, we can take $\mu^0 = 0$, which we shall do for the remainder of this paper. As noted in Zolotarev (1986), the functions $\sigma(\mathbf{t})$, $\beta(\mathbf{t})$, and $\mu(\mathbf{t})$ determine the distribution, an easy way to see this is that they determine $I_{\mathbf{X}}(\cdot)$:

$$I_{\mathbf{X}}(t) = \begin{cases} \sigma^\alpha(\mathbf{t})(1 - i\beta(\mathbf{t})\tan\frac{\pi\alpha}{2}) & \alpha \neq 1 \\ \sigma(\mathbf{t})(1 - i\mu(\mathbf{t})) & \alpha = 1. \end{cases}$$

2 Discrete spectral measures

In this section we discuss the case when Γ is a discrete spectral measure with a finite number of point masses, i.e.,

$$\Gamma(\cdot) = \sum_{j=1}^{n} \gamma_j \delta_{\mathbf{s}_j}(\cdot), \tag{4}$$

where γ_j's are the weights, and $\delta_{\mathbf{s}_j}$'s are point masses at the points $\mathbf{s}_j \in S_d, j = 1, \ldots, n$. Such spectral measures arise naturally in several cases: when the components of $\mathbf{X}$ are independent, when $\mathbf{X}$ arises from the finite dimensional distributions of a stable Ornstein-Uhlenbeck process, when one estimates Γ from data, etc. Discrete spectral measures form a particularly simple class to study, we now explain that they are "dense".

For a discrete spectral measure (4), (1) takes the form

$$\phi^*(\mathbf{t}) = \exp\left(-\sum_{j=1}^{n} \psi_\alpha(<\mathbf{t}, \mathbf{s}_j>)\,\gamma_j\right). \tag{5}$$

This expression is numerically simple, whereas calculation of $\phi(\mathbf{t})$ is generally difficult to compute. Let p be the density with characteristic function (1) and let p^* be the density corresponding to (5).

Theorem 1 *(Theorem 1, Byczkowski, Nolan and Rajput (1993)) Given $\epsilon > 0$, there exists an $n = n(d, \alpha, \epsilon, \Gamma)$, and values $\mathbf{s}_1, \ldots, \mathbf{s}_n$ and $\gamma_1, \ldots, \gamma_n$ so that*

$$\sup_{\mathbf{x} \in \mathbb{R}^d} |p(\mathbf{x}) - p^*(\mathbf{x})| < \epsilon. \tag{6}$$

We note that the proof gives a concrete bound for n and a way to choose the point masses. It also generalizes to approximating the probability of arbitrary sets, i.e. $\sup|P(A) - P^*(A)| < \epsilon$, where the supremum is over all Borel sets $A \subset \mathbb{R}^d$. Because of this approximation, little is lost by restricting ourselves to discrete spectral measures in what follows.

For our purposes, this approximation is useful for calculating $p(\mathbf{x})$ pointwise. With a bit more work, one can show that p and p^* are close in any $L^p(\mathbb{R}^d, d\mathbf{x})$, $1 \leq p < \infty$ if the discrete measure is constructed appropriately. Furthermore the supports of p and p^* can be guaranteed to be the same. (In the symmetric case or if $\alpha \geq 1$, then the support of a stable density is the span of the support of its spectral measure. In the nonsymmetric case when $\alpha < 1$, the support of a stable density is the cone generated by the support of its spectral measure, i.e. positive linear combinations of the support of the spectral measure. In all cases, careful choice of the $\mathbf{s}_1, \ldots, \mathbf{s}_n$ will guarantee identical support for p and p^*.)

3 Stable densities

An important result in inderstanding the connection between the spectral measure Γ and the distribution of $\mathbf{X}$ is the following theorem on the tails of the distribution due to Araujo and Gine (1980).

For a set $A \subset S_d$, define the cone generated by A to be $\text{Cone}(A) = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| > 0, \mathbf{x}/|\mathbf{x}| \in A\}$ $= \{r\mathbf{a} : r > 0, \mathbf{a} \in A\}$.

Theorem 2 *(Corollary 6.20, Araujo and Gine, 1980)*

$$\lim_{r\to\infty} \frac{P(\mathbf{X} \in \mathit{Cone}(A), |\mathbf{X}| > r)}{P(|\mathbf{X}| > r)} = \frac{\Gamma(A)}{\Gamma(S_d)}.$$

In words, the mass that Γ assigns to A determines the tail behavior of $\mathbf{X}$ in the "direction" A. In contrast, the local behavior of the distribution near the mode is mostly determined by the two quantities

$$I_0 = \inf\{\Re I(\mathbf{t}) : t \in S_d\} \qquad I_1 = \sup\{\Re I(\mathbf{t}) : t \in S_d\}.$$

In particular, I_0 and I_1 are invariant under rotations of the spectral measure, so local behavior is very different from the directional tail behavior in Theorem 2. The justification of these last remarks are contained in Abdul-Hamid (1996).

To understand more about multivariate stable densities, numerical methods can be used. The inversion formula for characteristic functions shows that

$$\begin{aligned} p(boldx) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i<\mathbf{x},\mathbf{t}>} e^{-I_{\mathbf{X}}(\mathbf{t})} d\mathbf{t} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i[<\mathbf{x},\mathbf{t}>+\Im I_{\mathbf{X}}(\mathbf{t})]-\Re I_{\mathbf{X}}(\mathbf{t})} d\mathbf{t} \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \cos[<\mathbf{x},\mathbf{t}> + \Im I_{\mathbf{X}}(\mathbf{t})] e^{-\Re I_{\mathbf{X}}(\mathbf{t})} d\mathbf{t} \end{aligned}$$

In Nolan and Rajput (1994), the problem of numerically evaluating these integrals is addressed. Here we present two examples of such calculations in two dimensions with a discrete spectral measure. More examples can be found in Byczkowski, Nolan and Rajput (1993) and in Nolan and Rajput (1994). Figure 1 shows a symmetric stable density with $\alpha = 0.9$ and $n = 6$ point masses: $\gamma_1 = 0.25$ at $\mathbf{s}_1 = (1, 0)$, $\gamma_2 = 0.125$ at $\mathbf{s}_2 = (1/2, \sqrt{3}/2)$, $\gamma_3 = 0.25$ at $\mathbf{s}_3 = (-1/2, \sqrt{3}/2)$, $\gamma_4 = 0.25$ at $\mathbf{s}_4 = (-1, 0)$, $\gamma_5 = 0.125$ at $\mathbf{s}_5 = (-1/2, -\sqrt{3}/2)$, $\gamma_6 = 0.25$ at $\mathbf{s}_6 = (1/2, -\sqrt{3}/2)$, Figure 2 shows a nonsymmetric stable density with $\alpha = 1.6$ and $n = 5$ point masses: $\gamma_1 = 0.1$ at $\mathbf{s}_1 = (1, 0)$, $\gamma_2 = 0.3$ at $\mathbf{s}_2 = (\sqrt{3}/2, 1/2)$, $\gamma_3 = 0.1$ at $\mathbf{s}_3 = (1/2, \sqrt{3}/2)$, $\gamma_4 = 0.3$ at $\mathbf{s}_4 = (0, 1)$, $\gamma_5 = 0.1$ at $\mathbf{s}_5 = (-1/2, \sqrt{3}/2)$.

While the densities are unimodal, their fine behavior can be very different from a Gaussian distribution. In the symmetric case, the mode is at the origin, in the nonsymmetric case the mode is shifted. The exact location of the mode is not known in general; a necessary step in finding the mode is to find the mode of a one dimensional stable density, which is still not exactly known. Applied to a discrete measure, Theorem 2 says that the density will have "creases" along the rays starting at the origin and passing through the point masses. The level curves show this difference more distinctly than the density surface. The level curves are star-like in Figure 1, while they are egg-shaped in Figure 2. Generally, as α gets closer to 2 and as the mass of Γ is more uniformly spread, the level curves get rounder.

Recently, we have derived a new formula for multivariate stable densities that is described in Abdul-Hamid and Nolan (1996).

4 Simulation of stable random vectors

There are several reasons to simulate stable random vectors. One is to have a source of multivariate noise with heavy tails to use in evaluating the robustness of multivariate statistical methods. A second reason is to be able to use Monte Carlo techniques. The numerical computations discussed in the previous section are computationally expensive. The availability of a fast method to generate stable random variables makes it possible to empirically estimate a stable density or the probability content of a set quickly. Currently, it is only feasible to numerically calculate two dimensional densities, and the computation time is a complicated function of α and the number, mass and spread of the point masses. In any dimension, the computation time for simulation is essentially a linear

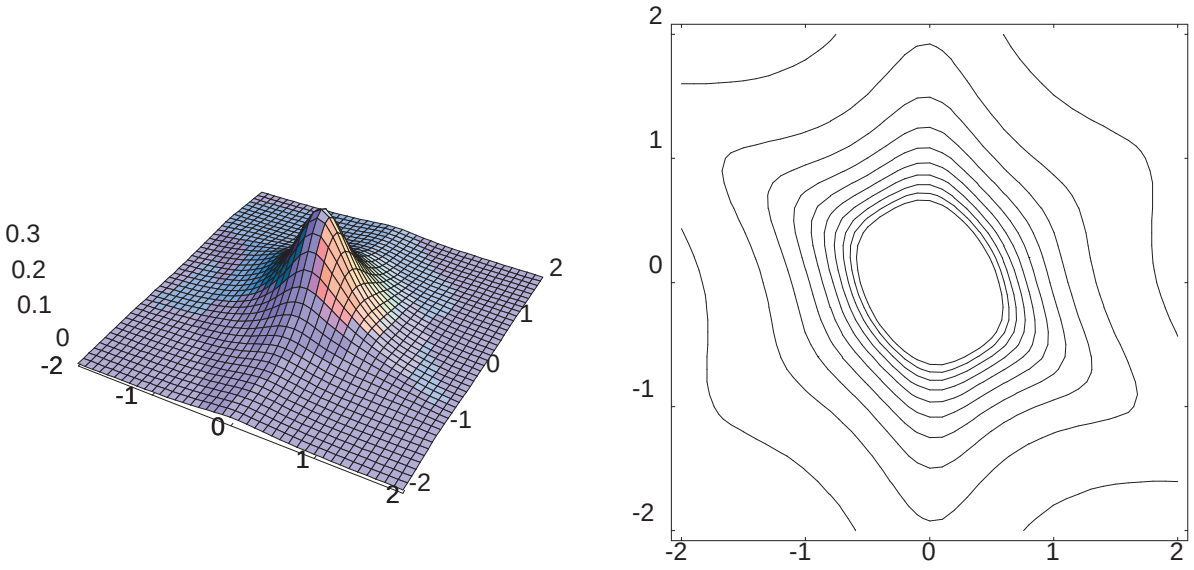

Figure 1: Two dimensional stable density surface and level curves with $\alpha = 0.9$.

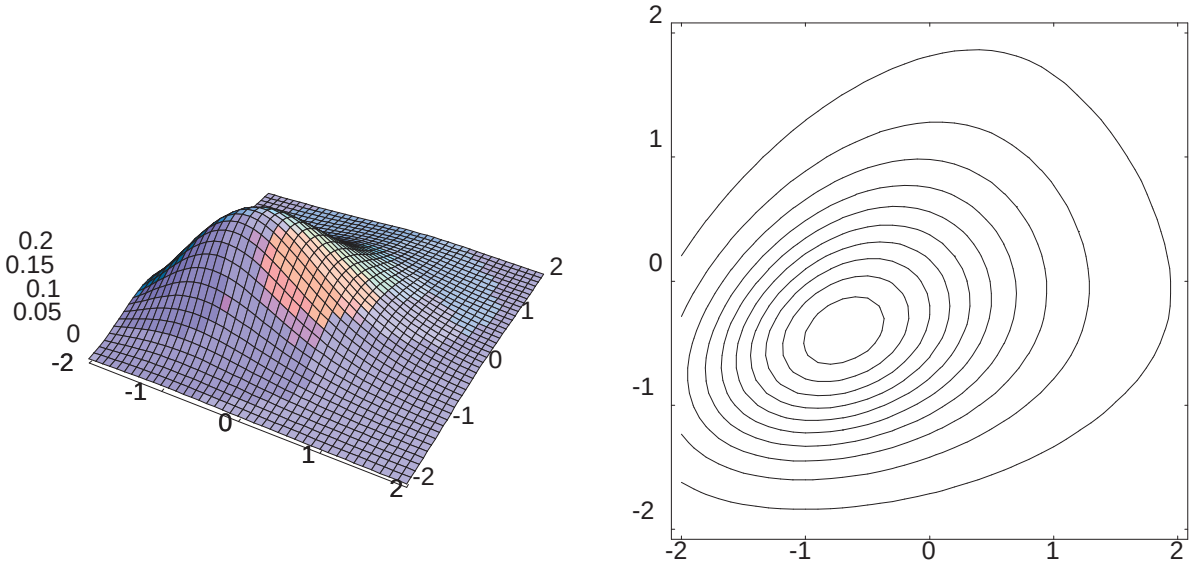

Figure 2: Two dimensional stable density surface and level curves with $\alpha = 1.6$.

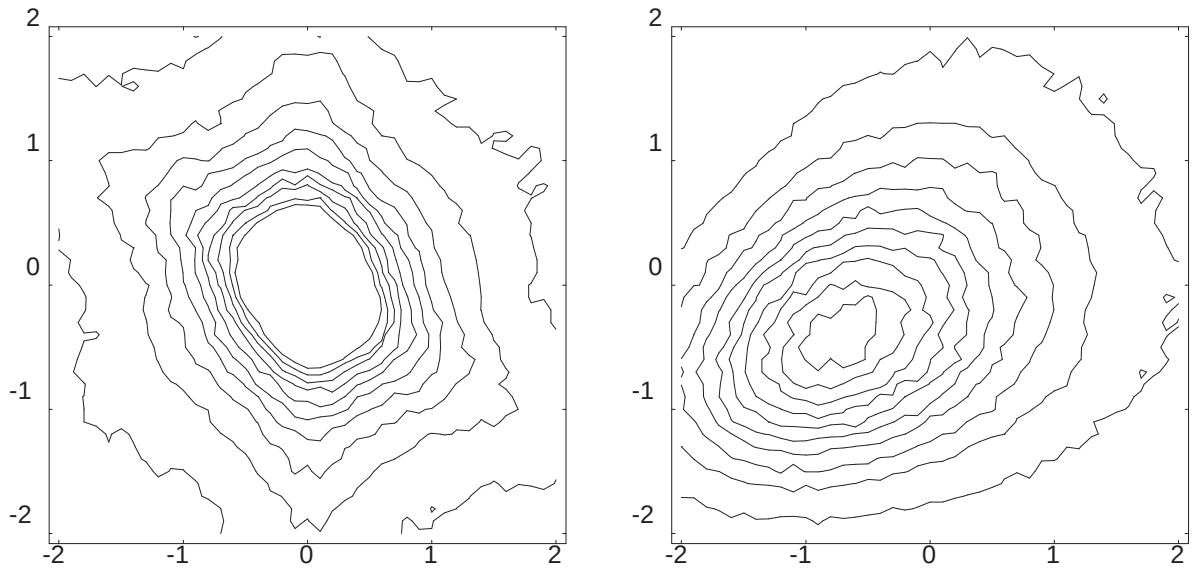

Figure 3: Contour plots of empirical densities from simulation of two dimensional stable distributions. On the left is the distribution shown in Figure 1, on the right is the distribution shown in Figure 2. 1,000,000 simulations were done in each case.

function of the number of point masses only. (Of course in higher dimensions, many simulations are necessary to get accurate estimates.)

The method is based on the following result from Modarres and Nolan (1994). If $\mathbf{X}$ has characteristic function (5), then

$$\mathbf{X} \stackrel{D}{=} \begin{cases} \sum_{j=1}^{n} \gamma_j^{1/\alpha} Z_j \mathbf{s}_j & \alpha \neq 1 \\ \sum_{j=1}^{n} \gamma_j (Z_j + \frac{2}{\pi} \ln \gamma_j) \mathbf{s}_j & \alpha = 1, \end{cases} \tag{7}$$

where $Z_1, \ldots, Z_n$ are iid totally skewed, standardized one dimensional α-stable random variables, i.e. $Z_i \sim S_\alpha(1, 1, 0)$. (When $\mu^0 \neq 0$, both cases above have a $+\mu^0$ in them.)

Figure 3 shows tabulated results of simulations using the spectral measures described in Figures 1 and 2. Computation times on a 60 MHz Pentium for Figures 1 and 2 were around 30 minutes each, whereas the simulations in Figure 3 took approximately 10 minutes each.

5 Estimation of stable spectral measures

The need for an estimator of a multivariate α-stable spectral measure arises in connection with stochastic modeling of financial portfolios. The spectral measure carries essential information about the vector, in particular about the dependence structure between the individual stocks that make up the portfolio. Other applications involve modeling the relation between exchange rates of different currencies. In Tsakalides and Nikios (1995), it is shown that modeling data from an array of radar sensors with a two dimensional (radially symmetric) Cauchy distribution yields significant improvement over the standard Gaussian model. Analysis of such experimental data may give a better model for the data. Finally, combined with Section 6, methods of estimating a spectral measure give a way of identifying stable processes.

One solution to this estimation problem is due to Rachev and Xin (1993) and Cheng and Rachev (1995). In Nolan, Panorska and McCulloch (1996), two other solutions are given; all 3 methods are implemented and compared on test data. Below we give a brief description of all three methods; for conciseness we assume that $\mu^0 = 0$.

The Rachev-Xin-Cheng method is based on Theorem 1. An ad hoc value is picked for r and it is used to estimate the measure of as set $A \subset S_d$ by:

$$\widehat{\Gamma}(A) = \text{const.} \frac{\#\{\mathbf{X}_i : |\mathbf{X}_i| > r, \mathbf{X}_i \in \text{Cone}(A)\}}{\#\{\mathbf{X}_i : |\mathbf{X}_i| > r\}}.$$

The next two estimators are based on using the sample to estimate the ch. f. on some grid. More precisely, we will estimate $I_{\mathbf{X}}(\cdot)$ on a grid $\mathbf{t}_1, \ldots, \mathbf{t}_n \in S_d$. Below we show how to recover an estimate of the spectral measure from this.

The empirical characteristic function (ECF) method is straightforward. Given an i.i.d. sample $\mathbf{X}_1, \ldots, \mathbf{X}_k$ of α-stable random vectors with spectral measure Γ, let $\widehat{\varphi}_k(\mathbf{t})$ and $\widehat{I}_k$ be the empirical counterparts of ϕ and I, i.e. $\widehat{\varphi}_k(\mathbf{t}) = (1/k)\sum_{j=1}^{k} \exp(i < \mathbf{t}, \mathbf{X}_j >)$ is the sample characteristic function, and $\widehat{I}_k(\mathbf{t}) = -\ln \widehat{\varphi}_k(\mathbf{t})$. Given a grid $\mathbf{t}_1, \ldots, \mathbf{t}_n \in S_d$, $\vec{I}_{ECF,k} = \left[\hat{I}_k(\mathbf{t}_1), \ldots, \hat{I}_k(\mathbf{t}_n)\right]'$ is the ECF estimate of $I_{\mathbf{X}}(\cdot)$.

The last method was defined by McCulloch (1994); we call it a projection method because it is based on one-dimensional projections of the data set. For each gridpoint $\mathbf{t}_j$, define the one-dimensional data set $< \mathbf{t}_j, \mathbf{X}_1 >, \ldots, < \mathbf{t}_j, \mathbf{X}_k >$. Use some method to estimate the scale $\hat{\sigma}(\mathbf{t}_j)$ and skewness $\hat{\beta}(\mathbf{t}_j)$ (and shift $\hat{\mu}(\mathbf{t}_j)$ when $\alpha = 1$) of this one-dimensional data. The best method for doing this is to use maximum likelihood estimation. While several researchers have used this approach, it is prohibitively expensive (in computational time) in general. Efficient programs exist for calculating symmetric stable densities for $\alpha > 0.85$ and new programs will soon be available for calculating general stable densities, so this method will become practical soon. See McCulloch (1995) for references to ML estimation and the symmetric density program, see Nolan (1996a) for the nonsymmetric case. Since we needed a fast method for estimating parameters of general stable distributions, we used the quantile based estimators of McCulloch (1986) in the examples below. Define

$$\hat{I}_k(\mathbf{t}_j) = \begin{cases} \hat{\sigma}^{\alpha}(\mathbf{t}_j)\left(1 - i\hat{\beta}(\mathbf{t}_j)\tan\frac{\pi\alpha}{2}\right) & \alpha \neq 1 \\ \hat{\sigma}(\mathbf{t}_j)\left(1 - i\hat{\mu}(\mathbf{t}_j)\right) & \alpha = 1 \end{cases}$$

The vector $\vec{I}_{PROJ,k} = \left[\hat{I}_k(\mathbf{t}_1), \ldots, \hat{I}_k(\mathbf{t}_n))\right]'$ is the projection estimator of $I_{\mathbf{X}}(\cdot)$.

In order to obtain an estimate of the spectral measure $\widehat{\Gamma}$, we need to invert the discrete approximations to the characteristic function obtained by the emprical ch. f. method and by the projection method. We start with the case when Γ is a discrete spectral measure of form (4). Let $I_{\mathbf{X}}(\mathbf{t}) = \sum_{j=1}^{n} \psi_\alpha(< \mathbf{t}, \mathbf{s}_j >)\gamma_j$. Further, let $\mathbf{t}_1, \ldots, \mathbf{t}_n \in R^d$, and define the $n \times n$ matrix

$$\Psi = \Psi(\mathbf{t}_1, \ldots, \mathbf{t}_n; \mathbf{s}_1, \ldots, \mathbf{s}_n) = \begin{bmatrix} \psi_\alpha(< \mathbf{t_1}, \mathbf{s_1} >) & \ldots, & \psi_\alpha(< \mathbf{t_1}, \mathbf{s_n} >) \\ \vdots & \vdots & \vdots \\ \psi_\alpha(< \mathbf{t_n}, \mathbf{s_1} >) & \ldots, & \psi_\alpha(< \mathbf{t_n}, \mathbf{s_n} >) \end{bmatrix}.$$

If $\vec{\gamma} = [\gamma_1, \ldots, \gamma_n]'$, and $\vec{I} = [I_{\mathbf{X}}(\mathbf{t}_1), \ldots, I_{\mathbf{X}}(\mathbf{t}_n)]'$, then

$$\vec{I} = \Psi\vec{\gamma}. \tag{8}$$

If $\mathbf{t}_1, \ldots, \mathbf{t}_n \in R^d$ are chosen so that Ψ^{-1} exists, then $\vec{\gamma} = \Psi^{-1}\vec{I}$ is the exact solution of (8).

For a general spectral measure Γ (not discrete and/or the location of the point masses are unknown), consider a discrete approximation $\Gamma^* = \sum_{j=1}^{n} \gamma_j \delta_{\mathbf{s_j}}$, where $\gamma_j = \Gamma(A_j)$, $i = 1, \ldots, n$ are the weights, and $\delta_{\mathbf{s}_j}$'s are point masses. When $d = 2$, it is natural to take $\mathbf{s}_j = (\cos(2\pi(j-1)/n), \sin(2\pi(j-1)/n)) \in S_d$, and arcs $A_j = (2\pi(j - (3/2))/n, 2\pi(j - (1/2))/n]$, $j = 1, \ldots, n$. In higher dimensions, the A_j's are patches that partition the sphere S_d, with some "center" $\mathbf{s}_j$. In this case, each of the coordinates of the $\vec{\gamma} = [\gamma_1, \ldots, \gamma_n]'$ is an approximation to the mass of the patch containing $\mathbf{s}_j$, $j = 1, \ldots, n$.

The principle behind the estimation of Γ is simple: given some grid $\mathbf{t}_j = \mathbf{s}_j$, $j = 1, \ldots, n$ and either estimate ($\vec{I}_{ECF}$ or $\vec{I}_{PROJ}$) of $\vec{I}$, invert (8) to get $\vec{\gamma}$. Using these weights and the grid

$\mathbf{s}_1, \ldots, \mathbf{s}_n$, define $\hat{\Gamma}$ by (4). While the above method is formally correct, it has several numerical problems. The paper of Nolan, Panorska and McCulloch (1996) restates this problem as the solution to a real valued constrained quadratic programming problem to deal with the complex values and ill-conditioned nature of the Ψ matrix.

5.1 Examples of estimating Γ

We shall consider two examples of estimating the spectral measure for bivariate data. In the first one, the data is simulated using the procedure of Section 4, the second one estimates the spectral measure for a real sample consisting of foreign exchange rates. The empirical characteristic function method and projection methods with quadratic programming solution were used, the procedure of Rachev-Xin-Cheng (RXC) was also used to recover spectral measures. In all cases, we used 100 evenly spaced gridpoints $\mathbf{t}_j, j = 1, \ldots, 100$, where we estimated the weights γ_j. Both examples took less than a minute of execution time on a fast PC. We turn to the description of examples now.

Example 1. Data was simulated from the measure $\Gamma_1(\cdot) = \sum_{j=1}^{3}(1/3)\delta_{\mathbf{s}_j}(\cdot)$, where $\mathbf{s}_j = (\cos\theta_j, \sin\theta_j) \in S_d, j = 1, \ldots, 3$, and $\theta_1 = \pi/3, \theta_2 = \pi,$ and $\theta_3 = 3\pi/2$. The shape parameter is $\alpha = 1.25$. The estimated shape parameters for the three methods were: $\alpha_{RXC} = 1.383$, $\alpha_{PROJ} = 1.275$, and $\alpha_{ECF} = 1.270$. The results of estimation of the cumulative spectral measure are shown in Figure 4. Both the ECF and PROJ methods do a good job of recovering Γ - both the location and the size of the point masses are accurately determined. The RXC method seriously underestimates the total mass, smooths out the mass instead of identifying point masses and overestimates α.

Example 2. We looked at a portfolio of two assets: German Mark vs US Dollar (DMUS), and Japanese Yen vs US Dollar (JYUS) exchange rates. The data is in the form of daily log-returns, and spans the period from January 1, 1980 to December 7, 1990 (total of 2563 values). We plotted the results of estimation for this data in Figure 5. The estimated shape parameters for the three methods were: $\alpha_{RXC} = 1.579$, $\alpha_{PROJ} = 1.531$, and $\alpha_{ECF} = 1.529$.

6 Identification of stable processes

If a physical process is observed to have stable finite dimensional distributions, it may be of interest to know what type of process is being observed. For example, is it a moving average process, harmonizable, or self-similar? This section will give a brief discussion of how to relate the stochastic integral representation of a stable process to the spectral measure of finite dimensional distributions. By estimating the spectral measure using the preceeding section, we may be able to identify the type of process. Since we will be estimating spectral masures from data, it is unlikely that we can precisely identify which type of process is being observed, rather we can identify plausible models and rule out implausible ones

An alternative way of representing $\mathbf{X}$ in distribution is the stochastic integral representation, see Theorem 3.5.6 of [14]:

$$\mathbf{X} \stackrel{D}{=} \int_M \mathbf{f}(u)W(du) = \left(\int_M f_1(u)W(du), \ldots, \int_M f_d(u)W(du)\right), \tag{9}$$

where W is an α-stable measure defined on some measure space $(M, \mathcal{M}, m)$ and $\mathbf{f} = (f_1, \ldots, f_d)$ is a vector of $\mathcal{M}$-measurable funtions. (Again we take $\mu^0 = 0$.)

A third way of representing $\mathbf{X}$ is in terms of a series representation originally due to Lepage, see Theorem 3.10.1 in [14]. For our purposes, this series representation is equivalent to the stochastic integral representation in that it is parameterized by $\mathbf{f}$ and the measure space $(M, \mathcal{M}, m)$.

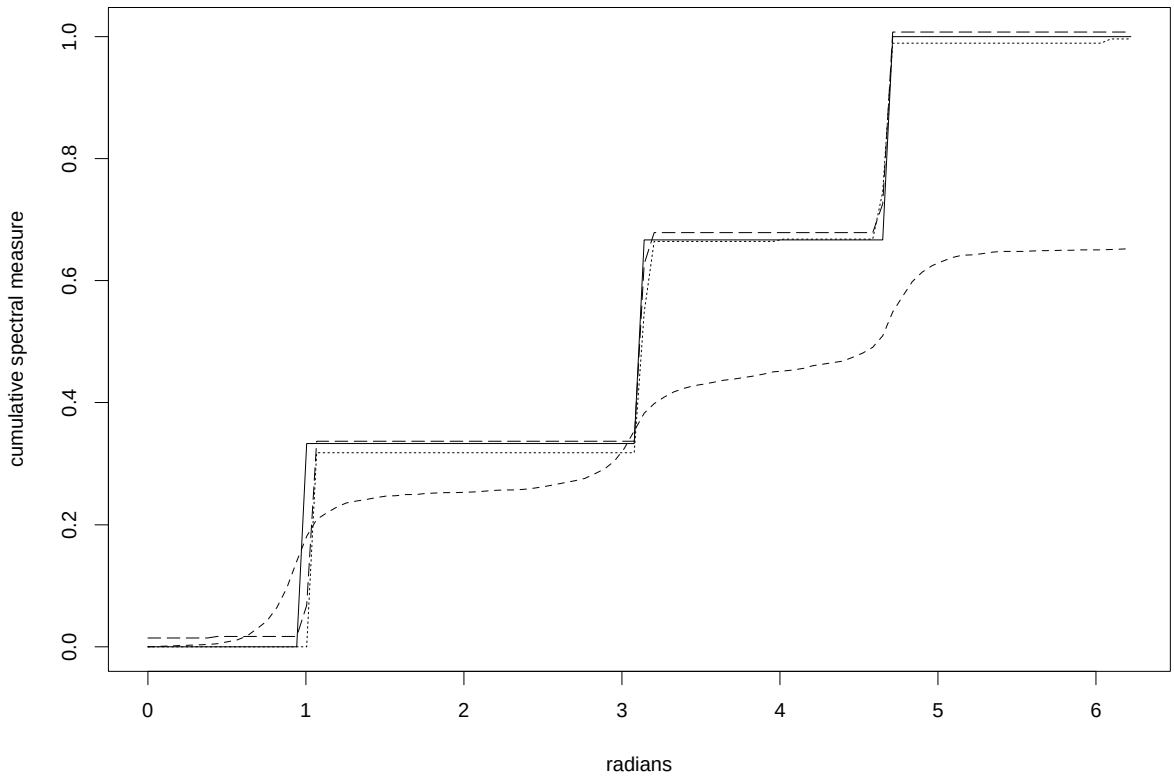

Figure 4: Estimates of the cumulative spectral measure from simulated data. The solid line is the exact cumulative, short dashed line is the Rachev-Xin-Cheng estimator, dotted line is the projection estimator, long dashed line is the characteristic function estimator. (10,000 data vectors)

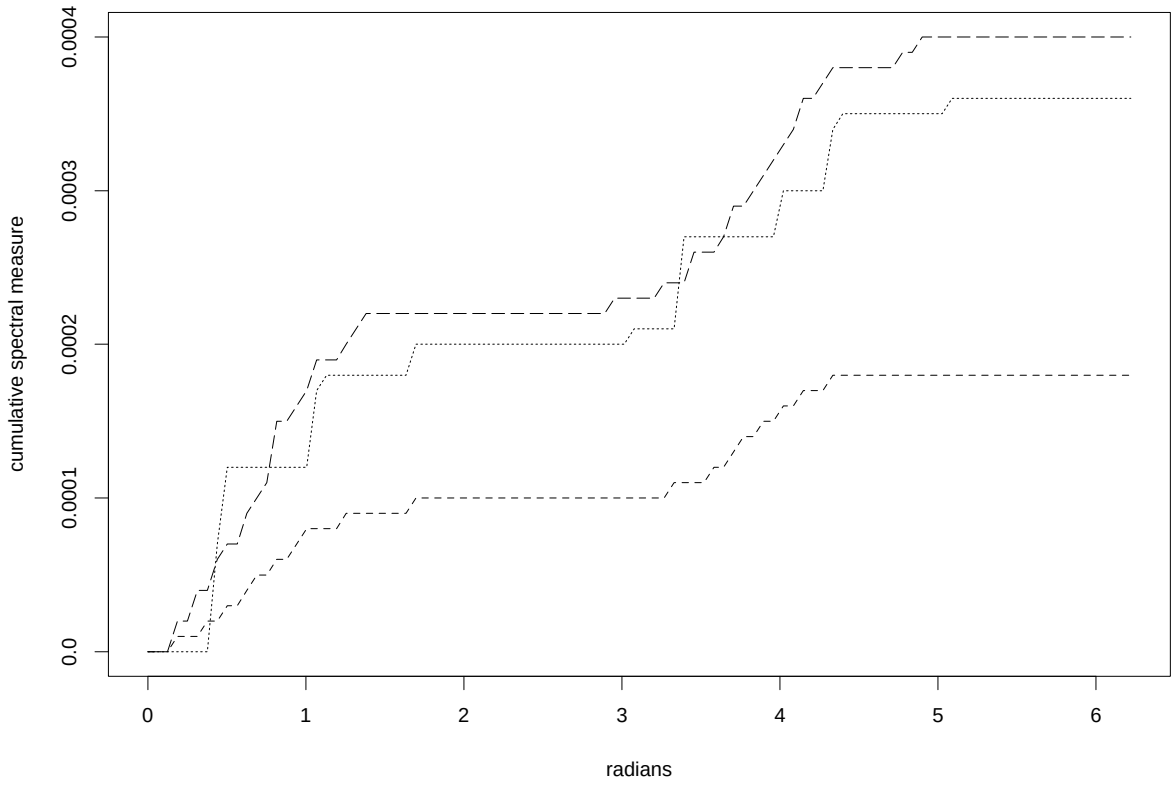

Figure 5: Estimates of the cumulative spectral measure corresponding to the exchange rates data. Line types are as in Figure 4. (2,563 data vectors)

Theorem 3 *(Samorodnitsky and Taqqu (1994), Nolan (1996b)) The spectral measure corresponding to the stochastic integral (9) is given by:*

$$\Gamma(A) = \int_{\{x \in M: \mathbf{f}(x) \in Cone(A)\}} |\mathbf{f}(x)|^{\alpha} \, m(dx),$$

where A is any Borel subset of S_d.

Note that if the random measure W is symmetric, then $\mathbf{X}$ will always be symmetric, but the measure Γ given by Theorem 3 may not be symmetric. It does give the right integral in (2), if we use the symmetric kernel function $\psi_\alpha(u) = |u|^\alpha$. The (unique) symmetric measure that will yield (2) is $\Gamma^{sym}(A) \equiv (\Gamma(A) + \Gamma(-A))/2$.

6.1 Examples

Consider the real part of a harmonizable process, i.e. a process of the form $X(t) = \Re \int_{\mathbb{R}} e^{itu} W(du)$, $t \in \mathbb{R}$, where W is a complex valued, rotationally invariant α-stable random measure determined by a finite Borel measure m on $\mathbb{R}$. The distribution of $X(t)$ is determined completely by m. The rotational symmetry of W means that $X(t)$, the real part of the stochastic integral defined above, is given by $X(t) = k(\alpha) \int_{\mathbb{R}} \cos(tu) W_1(du)$, where W_1 is a real symmetric random measure with the same control measure m and $k(\alpha) = \left(\frac{1}{2\pi} \int_0^{2\pi} |\cos\theta|^\alpha d\theta \right)^{1/\alpha}$.

The next theorem gives an explicit formula for the spectral measure of the two dimensional distributions of such processes. For the statement, define $\forall \theta \in (-\pi/4, \pi/4), \forall h \in \mathbb{R}, B_h(\theta) = \{x \in \mathbb{R} : \cos(hx) \leq \tan\theta\}$, and $Sector(\theta_1, \theta_2) = \{(\cos\theta, \sin\theta) : \theta_1 \leq \theta \leq \theta_2\}$.

Theorem 4 *Let Γ be the spectral measure of $(X(t), X(t+h))$ from a real harmonizable process with control measure m. Then support $(\Gamma) \subset Sector(-\pi/4, \pi/4) \bigcup Sector(3\pi/4, 5\pi/4)$ and for any $\theta \geq \pi/4$,*

$$\Gamma\left(Sector(-\pi/4, \theta)\right) = k(\alpha)^\alpha \int_{B_h(\theta)} \left(1 + \cos^2 hx\right)^{\alpha/2} m(dx).$$

We close with some numerical examples. Figure 6 shows the spectral measure of $(X(t), X(t+h))$ where $X(t)$ is a harmonizable process. By stationarity, we can assume $t = 0$, in which case $\mathbf{f}(x) = (1, \cos(hx))$ on $M = \mathbb{R}$. The right graph shows a polar plot of the spectral measure. For reference, the inner circle is the unit circle and the outer curve is a plot of $r(\mathbf{s}) = 1 + c\Gamma(ds)$. (The constant c was choosen for visual display purposes.) The last figure corresponds to a moving average process $X(t) = \int_{-\infty}^{t} g(x-t) W(dx)$, where W is the standard Lévy process (m is Lebesgue measure). Both examples used $m = 200$ points spread uniformly around the unit circle. More theoretical and numerical results are contained in Nolan (1996b).

References

[1] Abdul-Hamid, H. (1996) "Approxmiation of multivariate stable densities," PhD Dissertation, American University.

[2] Abdul-Hamid, H. and Nolan, J. P. (1996) Multivariate stable densities. Preprint.

[3] Araujo, A. and Giné, E. (1980) *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, NY.

[4] Byczkowski, T., Nolan, J. P., and Rajput, B. (1993), "Approximation of Multidimensional Stable Densities," *J. of Multiv. Anal.*, 46(1), 13-31.

[5] Cheng, B.N., and Rachev, S.T., (1995), "Multivariate stable securities in financial markets", *Mathematical Finance* (to appear).

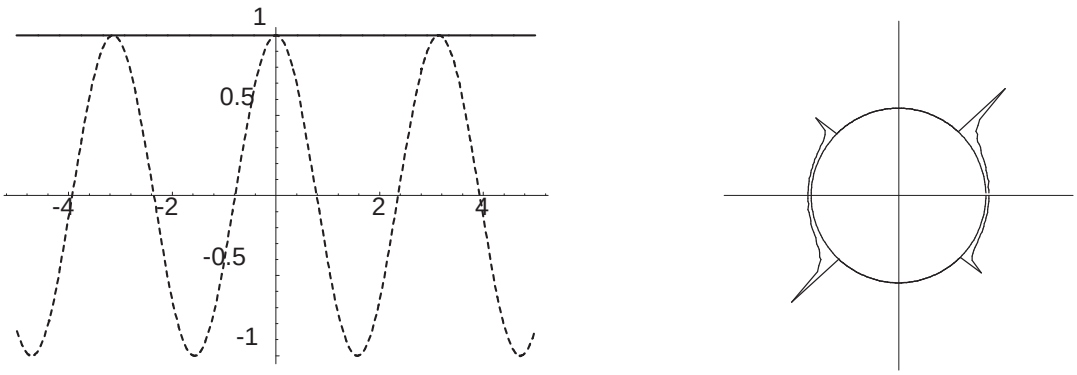

Figure 6: Spectral measure corresponding to $(X(t), X(t+h))$ for a real harmonizable stable process with $\alpha = 0.75$, $h = 2$, and spectral measure $m(dx) = 1/(1+|x|^3)dx$. On the left is the graph of $\mathbf{f}(x) = (1, \cos(hx))$ and on the right is a radial plot of the spectral measure.

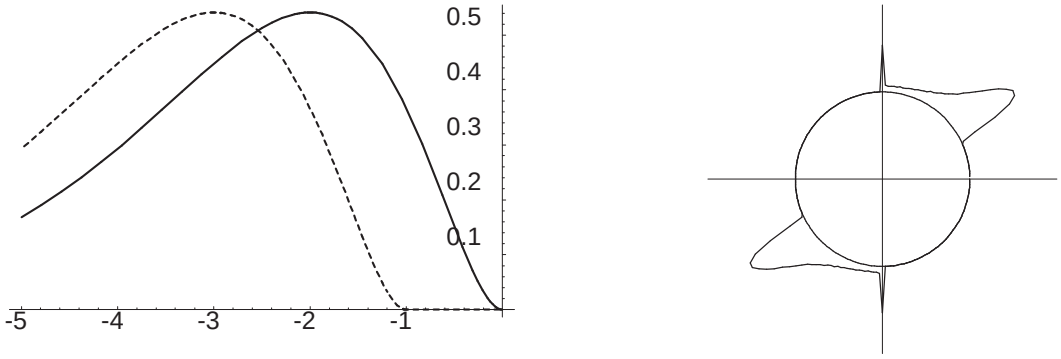

Figure 7: Spectral measure corresponding to $(X(t), X(t+h))$ for a moving average stable process with $\alpha = 1.5$, $h = 1$, and kernel function $g(x) = x^2 e^x 1_{(-\infty,0]}(x)$. On the left is the graph of $\mathbf{f}(x) = (g(x), g(x-h))$ and on the right is a radial plot of the spectral measure.

[6] McCulloch, J. H. (1986), "Simple Consistent Estimators of Stable Distribution Parameters," *Communications in Statistics. Simulation and Computation*, 15, 1109-1136.

[7] McCulloch, J. H. (1994), "Estimation of bivariate stable spectral densities," Unpublished manuscript, Department of Economics, Ohio State Univertsity.

[8] Modarres, R., and Nolan, J. P., (1994), "A Method for Simulating Stable Random Vectors", *Computational Statistics*, **9**, 11-19.

[9] Nolan, J. P. and Rajput, B., (1995) "Calculation of multidimensional stable densities", Commun. Statist. - Simula., **24**: 551-556.

[10] Nolan, J. P. (1996a), "An algorithm for computing one dimensional stable densities", In preparation.

[11] Nolan, J. P. (1996b), "Identification of stable measures", In preparation.

[12] Nolan, J. P., Panorska, A. K., McCulloch, J. H. (1996). Estimation of stable spectral measures. To appear.

[13] Rachev S.T., and Xin, H., (1993), " Test for Association of Random Variables in the Domain of Attraction of Multivariate Stable Law", *Probability and Mathematical Statistics*, **14** (I) 125 - 141.

[14] Samorodnitsky, G. and Taqqu, M. S. (1994) *Stable Non-Gaussian Random Processes*. Chapman and Hall, NY, NY.

[15] Tsakalides, P. and Nikias, C. (1995) Maximum likelihood localization of sources in noise modeled as a stable processes. *IEEE Trans. on Signal Proc.* **43:**

[16] Zolotarev, V. M. (1986) *One-dimensional Stable Distributions, Amer. Math. Soc. Transl. of Math. Monographs, Vol. 65*. Amer. Math. Soc., Providence, RI. (Transl. of the original 1983 Russian)