DATA ANALYSIS FOR HEAVY TAILED MULTIVARIATE SAMPLES

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ABSTRACT

In this paper we propose methods of exploratory data analysis that test for joint stability of a multivariate data set. Examples are shown of how these techniques support or reject the joint stability of the sample. If these methods suggest that a data set is jointly stable, we would like to know how well methods of estimating a spectral measure work. We examine how two previously described estimators of the stable spectral measure perform under a variety of assumptions.

1 INTRODUCTION

This paper is meant for practitioners who wish to draw conclusions from multivariate heavy tailed data sets. When faced with such a data set, tools
are needed to analyze the data and select appropriate models. We suggest methods for exploratory data analysis (EDA), estimation and goodness-of-fit procedures in the jointly stable case. After we discuss the methods, we give examples of their use, and comment on their performance. We hope that these ideas will be a useful contribution to the developing art of analyzing heavy tailed multivariate data sets.

The stable non-Gaussian distributions are commonly used for modeling heavy tailed phenomena. Their applications include models in finance and insurance (see Mandelbrot (1977), Fama (1965), Mittnik and Rachev (1993), Rachev and Samorodnitsky (1993), Jurlewicz, Weron and Weron (1996), Kim, Mittnik and Rachev (1996), Kozubowski and Rachev (1994), Panorska (1996), McCulloch (1996), Embrechts, Kluppelberg and Mikosch (1996)), physics (see Weron (1991), Weron and Jurlewicz (1993), Weron and Weron (1987), Scher, Shlesinger and Bendler (1991)), chemistry (see, Plonka and Paszkiewicz (1992), Pittel, Woyczynski and Mann (1990)), and biology (see Janicki and Weron (1994)) among others. For a thorough treatment of the theory and applications of stable distributions we refer the reader to the recent monographs by Samorodnisky and Taqqu (1993) and Janicki and Weron (1994). A substantial amount of work has been done on the estimation of the parameters of the univariate stable laws (see DuMouchel (1975), Hill (1975), Stuck (1976), McCulloch (1986), Kratz and Resnick (1996), Resnick and Stărică (1995), Nolan(1997b)). There is much left to be done however, for the multivariate stable case.

Our interest in this problem was motivated by applications of multivariate stable laws to modeling financial portfolios. The spectral measure carries essential information about the vector: it describes the dependence structure among the individual stocks that make up the portfolio. Thus there is interest in efficient and numerically sound estimators of a multivariate $\alpha$-stable spectral measure.

The major practical difficulty in dealing with stable laws, both univariate and multivariate, is that there are generally no closed forms for densities, and the only available information for stable random vectors is in the form of their characteristic functions. This lack of explicit densities causes difficulties or failure of many traditional methods of estimation, simulation, etc. However, some multivariate problems have been solved. Modarres and Nolan (1994) presented a method of simulating random samples from multivariate stable distributions. A method for approximation of a stable spectral measure by a discrete measure and (computationally intensive) methods of calculating multivariate stable densities were given in Byczkowski, Nolan, and Rajput (1993), and Nolan and Rajput (1995). Press (1972a, b) proposed a procedure for estimation of parameters and a definition of risk of a special type of stable portfolio. Rachev and Xin (1993) and Cheng and Rachev (1994) presented an
estimator of the stable spectral measure and investigated association problems for samples from the domain of attraction of a multivariate stable law. Their work inspired Nolan, Panorska and McCulloch (1996), who proposed two other estimators of the spectral measure for samples from a multivariate stable distribution. These were called the empirical characteristic function method (ECF) and the projection method (PROJ).

In this work we study numerical properties of the ECF and PROJ estimators proposed, and discuss some methods of the exploratory data analysis (EDA). Since the estimators assume that the sample is from a multivariate stable distribution, a practitioner should be able to verify that the sample is indeed multivariate stable before trying to estimate a spectral measure. Since we do not know of formal tests for this purpose, we propose some EDA tools that test the distributional assumptions.

In much of what follows, some method of estimating parameters of a one dimensional stable data set is needed. Ideally, one would use maximum likelihood estimation. There are now accurate programs to calculate stable densities; see McCulloch (1994b) for a symmetric density program, and Nolan (1997a) for a general (not necessarily symmetric) density program. While it is possible to do general maximum likelihood estimation for stable parameters it is still quite slow, see Nolan (1997b). Since we needed to estimate parameters of many one dimensional data sets, we used the fractile based method of estimating stable parameters due to McCulloch (1986) in our examples. While the approach below works for any $\alpha$, our current programs are limited to the range in which the McCulloch estimator works, namely $\alpha \geq 0.5$. Maximum likelihood estimation or any other method could be used with the techniques described below.

The paper is organized as follows. Section 2 proposes methods for the EDA of multivariate heavy-tailed data sets. Examples of the use of these methods to determine joint stability are presented on simulated data sets. In Section 3 we restate (for the sake of completeness) the definitions of the ECF and PROJ estimators of the spectral measure. Finally, we discuss aspects of the performance of these estimators in Section 4.

## 2 EXPLORATORY DATA ANALYSIS

We begin with preliminary definitions and notation. Let $\mathbf{X} \in \mathbb{R}^d$ be a multivariate $\alpha$-stable ($0 < \alpha < 2$) vector with characteristic function (ch.f.) (see Samorodnitsky and Taqqu (1994))

$$\phi_{\mathbf{X}}(t) = \mathbb{E}\exp\{i < \mathbf{X}, t >\} = \exp(-I_{\mathbf{X}}(t) + i < \mu, t >),$$
where the exponent function is given by
\[ I_X(t) = \int_{S^d} \psi_\alpha(<t,s>) \Gamma(ds) \]
Here \( S^d \) is the unit sphere in \( \mathbb{R}^d \), \( \Gamma \) is the spectral measure of the vector \( X \), \( \mu \) is a vector in \( \mathbb{R}^d \), \( <t,s> = t_1s_1 + \cdots + t_ds_d \) is the inner product, and
\[ \psi_\alpha(u) = \begin{cases} |u|^\alpha (1 - i \text{sign}(u) \tan \frac{\pi \alpha}{2}) & \alpha \neq 1 \\ |u|(1 + i \frac{\pi}{2} \text{sign}(u) \ln |u|) & \alpha = 1 \end{cases} \]
For a complex number \( z \), \( \Re z \) and \( \Im z \) denote the real and imaginary parts of \( z \).
For any \( t \in S^d \), the projection \( <t,X> \) of the random vector \( X \) on \( t \) is a one-dimensional stable random variable with characteristic function \( E \exp(iu <t,X>) = \exp(-I_X(ut)) \). Its scale, skewness and shift are given by (Example 2.3.4 of Samorodnitsky and Taqqu (1994)):
\[
\sigma^\alpha(t) = \Re I_X(t) = \int_{S^d} |<t,s>|^\alpha \Gamma(ds),
\]
\[
\beta(t) = \sigma^{-\alpha}(t) \int_{S^d} \text{sign} <t,s> |<t,s>|^\alpha \Gamma(ds)
\]
\[
= \begin{cases} -\Im I_X(t)/(\sigma^\alpha(t) \tan \frac{\pi \alpha}{2}) & \alpha \neq 1 \\ \Im [I_X(2t) - 2I_X(t)]/(4\sigma(t) \ln 2/\pi) & \alpha = 1 \end{cases}
\]
\[
\mu(t) = \begin{cases} 0 & \alpha \neq 1 \\ -\frac{\pi}{2} \int <t,s> \ln |<t,s>| \Gamma(ds) = -\Im I_X(t)/\sigma(t) & \alpha = 1 \end{cases}
\]
For a practitioner dealing with a sample from a heavy tailed distribution it is important to assess if the sample comes from a jointly stable distribution or not. We propose graphical EDA techniques for this assessment: qq-plots and density plots. Both the qq-plots and density plots were not practical until recently when methods of accurate numerical computation of stable densities, distribution functions and quantiles became available. These techniques are used in Nolan (1997b) for one dimensional data sets; here we explore their use on multi-dimensional data sets. The principle is straightforward: if the population is jointly \( \alpha \)-stable, then every one dimensional projection is univariate stable with the same \( \alpha \). (When \( \alpha \geq 1 \), this is a characterization of joint stability; when \( \alpha < 1 \), it is not - see §2.1 and 2.2 of Samorodnitsky and Taqqu (1994).)
Let \( X_1, \ldots, X_k \) be a sample of \( d \)-dimensional vectors. Pick a direction \( t \in S^d \) and form the one dimensional projection of the data \(<X_1,t>, \ldots, <X_k,t>\). Fit this one-dimensional sample with univariate stable distribution. For the given one dimensional data set, draw a qq-plot, i.e. plot quantiles of the data versus quantiles of the fitted stable distribution. If the data comes from the fitted stable distribution, then the graph should
be a straight line. A curved qq-plot signals departure of the projected data from stability, and hence the multivariate data is not jointly stable. This procedure should be repeated in several directions.

The density plot is a plot of the (smoothed) sample histogram and the fitted stable density. (The smoothed sample histogram is computed by using a Gaussian density as a convolution kernel to smooth the projection data.) You can easily see some departures from stability on a density plot, for example differences in heaviness of the tail, and multimodality of the projection.

Finally, we should compare estimated α’s from each projection to see if they are close. We do not know of formal procedures for testing equality of these (dependent) estimates, we only state that noticeable differences argue against joint stability.

2.1 Examples

We shall consider three examples of heavy tailed bivariate data. In all of the examples, the data was simulated using the Modarres and Nolan (1994) procedure.

Example 1. Data was simulated from a mixture of two stable distributions with independent components and common spectral measure $\Gamma_1(\cdot) = \sum_{j=0}^{3}(1/4)\delta_{s_j}(\cdot)$ with $s_j = (\cos j\pi/2, \sin j\pi/2)$, $j = 0, \ldots, 3$. The first distribution had $\alpha = 0.85$ and $\mu = (0, 0)$, the second had $\alpha = 1.4$ and $\mu = (1, 1)$. The mixing probability was 0.5 and the sample size was 5,000. We projected the data in several directions, with similar results. For illustration we include the graphs of the projection on 248.4 degrees in Figure 1. Both plots show departure from stability: the qq-plot is far from straight, and the density plot detects the bimodality of the bivariate data through this one dimensional projection.

Example 2. Data was simulated from a bivariate stable distribution with independent components, $\alpha = 1.2$, $\mu = (0, 0)$ and spectral measure $\Gamma_2 = \Gamma_1$ (from Example 1). Sample size was 10,000. We projected the data in several directions, and in all cases got a good fit to a stable distribution. For illustration, we plotted the EDA results for projections on 90 and 43.2 degrees in Figure 2.

Example 3. Data was simulated from a bivariate stable distribution with $\alpha = 1.5$, $\mu = (0, 0)$ and spectral measure $\Gamma_3(\cdot) = \sum_{j=0}^{19}(0.2)\delta_{s_j}(\cdot)$ with
Figure 1: Exploratory data analysis for a sample of 5,000 vectors from a mixture of stable distributions. LEFT: QQ-plot of the sample quantiles versus the quantiles of the fitted stable distribution. Solid line is the qq-plot. Dotted line is the plot of the straight line that shows nonlinearity of the qq-plot. RIGHT: Density plot. Solid line is the density of the fitted stable distribution. Dotted line is the smoothed histogram of the projected sample.

\[ s_j = (\cos j\pi/40, \sin j\pi/40) \in S^d, j = 0, \ldots, 19. \] Note that all the mass is concentrated in the first quadrant. Sample size was 10,000. We looked at numerous projections and saw close fits to univariate stable distributions. We show two projections: on 45 and 135 degrees in Figure 3. The graphs clearly show a distribution skewed to the right for the 45 degree projection, and a symmetric distribution for the 135 degree projection.

We note that the execution time for producing a qq-plot and a smoothed sample density vs. fitted density for one direction is approximately 15 seconds on a fast PC for a sample size of \( k = 10,000 \).

## 3 ESTIMATION PROCEDURES IN THE JOINTLY STABLE CASE

Once we are fairly confident that our data set comes from a multivariate distribution, we may proceed with the estimation of parameters. Given an iid sample \( X_1, \ldots, X_k \) of \( d \)-dimensional random vectors drawn from this distribution, we will first shift to eliminate \( \mu \), and then focus on the estimation of the spectral measure \( \Gamma \). To do the shift, estimate the one-dimensional parameters \( (\hat{\alpha}_j, \hat{\sigma}_j, \hat{\beta}_j, \hat{\mu}_j), j = 1, \ldots, d \) for each of the coordinates of the \( d \)-dimensional data set. Use the estimate \( \hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_d) \) for the shift vector and subtract it from the sample. From now on, we assume that the data has been shifted so that \( \mu \) is zero.

Both estimators are based on using the sample to estimate the exponent
Figure 2: Exploratory data analysis for a sample from $\Gamma_2$. TOP: QQ-plot and density plot for the projection on the y-axis. BOTTOM: QQ-plot and density plot for the projection in direction of 43.2 degrees angle. Line types are the same as in Figure 1.

of the ch. f. $I_X(\cdot)$ on a grid $t_1, \ldots, t_n \in S^d$. Then, they recover an estimate of the spectral measure from the estimate of $I_X(\cdot)$ and an estimate of $\alpha$.

The ECF method is quite intuitive. Given an i.i.d. sample $X_1, \ldots, X_k$ of $\alpha$-stable random vectors with spectral measure $\Gamma$, let $\hat{\Phi}_k(t)$ and $\hat{I}_k$ be the empirical counterparts of $\phi$ and $I$, i.e. $\hat{\Phi}_k(t) = (1/k) \sum_{i=1}^{k} \exp(i < t, X_i >)$ is the sample characteristic function, and $\hat{I}_k(t) = -\ln \hat{\Phi}_k(t)$. Given a grid $t_1, \ldots, t_n \in S^d$, $\hat{I}_{ECF,k} = \left[ \hat{I}_k(t_1), \ldots, \hat{I}_k(t_n) \right]'$ is the ECF estimate of $I_X(\cdot)$. Define $\hat{\alpha}_{ECF} = \left( \sum_{j=1}^{d} \hat{\alpha}_j \right) / d$ as the ECF estimator of the joint index of stability $\alpha$.

The projection (PROJ) method was first defined by McCulloch (1994a), and it is based on one-dimensional projections of the data set. For each gridpoint $t_j$, estimate the scale $\hat{\sigma}(t_j)$ and skewness $\hat{\beta}(t_j)$ (and shift $\hat{\mu}(t_j)$ when $\alpha = 1$) of the one-dimensional data set $< t_j, X_1 >, \ldots, < t_j, X_k >$. The pooled estimate of the joint index of stability for the projection method
is $\hat{\alpha}_{\text{PROJ}} = \left(\sum_{j=1}^{n} \hat{\alpha}(t_j)\right)/n$. Next define

$$
\hat{I}_k(t_j) = \begin{cases} 
\hat{\alpha}^{\text{PROJ}}(t_j) \left(1 - i\hat{\beta}(t_j) \tan \frac{\pi \hat{\alpha}_{\text{PROJ}}}{2}\right) & \hat{\alpha}_{\text{PROJ}} \neq 1 \\
\hat{\alpha}(t_j) \left(1 - i\hat{\mu}(t_j)\right) & \hat{\alpha}_{\text{PROJ}} = 1
\end{cases}
$$

(2)

The vector $\vec{I}_{\text{PROJ},k} = [\hat{I}_k(t_1), \ldots, \hat{I}_k(t_n)]'$ is the projection estimator of $I_X(\cdot)$.

We will consider the sample size $k$ as fixed and simplify notation by supressing the dependence of the estimators on $k$, e.g. write $\vec{I}_{\text{ECF}}$ for $\vec{I}_{\text{ECF},k}$, etc. In order to obtain an estimate $\hat{\Gamma}$ for the spectral measure $\Gamma$, we invert the discrete approximations to the characteristic function obtained by either the ECF or the PROJ method. First, let $\hat{\Gamma}$ be a discrete spectral measure, i.e.,

$$
\hat{\Gamma}(\cdot) = \sum_{j=1}^{n} \gamma_j \delta_{s_j}(\cdot),
$$

(3)

where $\gamma_j$’s are the weights, and $\delta_{s_j}$’s are point masses at the points $s_j \in$
\[ S^d, j = 1, \ldots, n. \] Let \( I_X(t) = \sum_{j=1}^{n} \psi_a(<t, s_j>) \gamma_j. \) Further, let \( t_1, \ldots, t_n \in R^d, \) and define the \( n \times n \) matrix
\[
\Psi = \Psi(t_1, \ldots, t_n; s_1, \ldots, s_n) = \begin{bmatrix}
\psi_a(<t_1, s_1>) & \cdots & \psi_a(<t_1, s_n>) \\
\vdots & \ddots & \vdots \\
\psi_a(<t_n, s_1>) & \cdots & \psi_a(<t_n, s_n>)
\end{bmatrix}.
\]

If \( \tilde{\gamma} = [\gamma_1, \ldots, \gamma_n]', \) and \( \tilde{I} = [I_X(t_1), \ldots, I_X(t_n)]' \), then
\[
\tilde{I} = \Psi \tilde{\gamma}.
\]

If \( t_1, \ldots, t_n \in R^d \) are chosen so that \( \Psi^{-1} \) exists, then \( \tilde{\gamma} = \Psi^{-1} \tilde{I} \) is the exact solution of (4).

For a general \( \Gamma \) (not discrete and/or the location of the point masses are unknown), consider a discrete approximation \( \Gamma^* = \sum_{j=1}^{n} \gamma_j \delta_{s_j}, \) where \( \gamma_j = \Gamma(A_j), i = 1, \ldots, n \) are the weights, and \( \delta_{s_j} \)'s are point masses. When \( d = 2, \) it is natural to take \( s_j = (\cos(2\pi(j - 1)/n), \sin(2\pi(j - 1)/n)) \in S^d, \) and arcs \( A_j = (2\pi(j - (3/2))/n, 2\pi(j - (1/2))/n) \), \( j = 1, \ldots, n. \) In higher dimensions, the \( A_j \)'s are patches that partition the sphere \( S^d, \) with some “center” \( s_j. \) In this case, each of the coordinates of \( \tilde{\gamma} = [\gamma_1, \ldots, \gamma_n]' \) is an approximation to the mass of the patch containing \( s_j, j = 1, \ldots, n. \)

Finally, given some grid \( t_j = s_j, j = 1, \ldots, n \) and either estimate \( \tilde{I}_{ECF} \) or \( \tilde{I}_{PROJ} \) of \( \tilde{I}, \) invert (4) to get \( \tilde{\gamma}. \) Using these weights and the grid \( s_1, \ldots, s_n, \) define \( \Gamma \) by (3). Unfortunately, there are numerical problems with this straightforward linear inversion scheme. In practice, we restate the problem as a constrained (because we restrict the point masses to be nonnegative) least squares problem and solve it using quadratic programming. For a discussion of the numerical implementation of this inversion problem, and some examples of spectral measure estimation from simulated and real samples from bivariate stable distributions we refer the reader to Nolan, Panorska and McCulloch (1996).

4 GRIDSIZE AND GOODNESS-OF-FIT

For both of the ECF and PROJ estimators, one needs to decide on the number of gridpoints used for estimating \( \Gamma. \) The fineness of the grid depends on the use you intend to make of the estimated spectral measure. If you are interested in examining the sample for independence or association, you only need to know roughly where the spectral mass is located. On the other hand, if you want to use the estimated spectral measure for detailed purposes, e.g. computing bivariate densities of the fitted distribution, then more precise location of the mass is needed. In either case, we would like to know how the estimators behave when the grid size is not optimal.
We do not have a formal procedure for choosing gridsize, but suggest another EDA procedure for assessing the goodness-of-fit between the sample and the fitted stable distribution corresponding to an estimated spectral measure with a given grid size. We initially wanted to compare the sample with the fitted distribution directly. This would involve a multivariate display of the sample and the fitted distribution, which has several difficulties. First, it is difficult to display the sample bivariate data - a straightforward histogram is too choppy and smoothing it requires estimating bandwidth parameters in both directions and is still likely to work poorly on the tails. Second, it is very computationally expensive to numerically compute the fitted density from the estimated spectral measure. Easton and McCulloch (1990) give a multivariate generalization of Q-Q plots based on matching. Unfortunately, it is not clear what a reliable "reference sample" should be with that method.

After some thought we decided on an alternative approach for comparing the sample data to the fitted distribution. For two stable random vectors \( \mathbf{X} \) and \( \mathbf{Y} \), Nolan (1997c) proposes a measure of closeness of stable distributions that compares the corresponding exponent functions \( I_{\mathbf{X}}(\cdot) \) and \( I_{\mathbf{Y}}(\cdot) \) on the sphere. We propose using a graphical comparison of exponent functions to compare the sample with the fitted distribution. The closeness will generally depend on the gridsize. Since the exponent functions are complex, it is not easy to compare them directly. We could look at the real and imaginary parts directly, but we suggest a variation of this that has more intuitive meaning: comparing the scale function \( \sigma(t) \) and the skewness function \( \beta(t) \) from (1). Equation (2) shows that, except when \( \alpha = 1 \), this is essentially the same as comparing the real and imaginary parts of the exponent function. These quantities have a concrete meaning as parameters of the projected one-dimensional data in each direction: \( \sigma(t) \) specifies the spread of the data and \( \beta(t) \) specifies the skewness of the data in the direction \( t \).

Data analysis yields three estimates of \( I_X(t) \): (1) the sample exponent from the data set \( \hat{I}(t) \), (2) the exponent corresponding to the spectral measure \( \hat{\Gamma}_{E CF} \) estimated by the ECF method \( \hat{I}_{E CF}(t) \), (3) the exponent corresponding to the spectral measure \( \hat{\Gamma}_{P RO J} \) estimated by the PROJ method \( \hat{I}_{P RO J}(t) \).

For a fixed grid \( t_1, \ldots, t_n \) on \( S^d \), let \( \hat{\beta}(t_j) \) and \( \hat{\sigma}(t_j) \) be the sample estimates of the skewness and scale. Next, let \( \hat{\beta}(t_j) \) and \( \hat{\sigma}(t_j) \) be the skewness and scale of the projections corresponding to an estimated spectral measure. We shall plot \( \hat{\beta}(t_j) \) together with \( \hat{\beta}(t_j) \) for both the ECF and PROJ estimators and \( \hat{\sigma}(t_j) \) together with \( \hat{\sigma}(t_j) \). If the estimated spectral measure describes the sample well, then the parameters estimated from the sample and those corresponding to the estimated spectral measure should be close. We suggest that one starts with a moderate grid, say 40 points, and increases the gridsize until \( \sigma \) and \( \beta \) estimated using PROJ and ECF methods fit the
Figure 4: Estimates of the spectral measure, $\sigma(\cdot)$ and $\beta(\cdot)$ for a sample of 10,000 stable vectors with independent components corresponding to $\Gamma_2$. LEFT column: Estimates for 100 gridpoints. RIGHT column: Estimates for 24 gridpoints. TOP: Polar graphs of the spectral measure. MIDDLE TOP: Cumulative spectral measure. MIDDLE BOTTOM: Estimates of $\sigma(\cdot)$. BOTTOM: Estimates of $\beta(\cdot)$. On all graphs dashed line corresponds to the ECF estimates and dotted line corresponds to the PROJ estimates. On the graphs of $\sigma(\cdot)$ and $\beta(\cdot)$, additional solid line corresponds to sample estimates.
Figure 5: Estimates of the spectral measure, $\sigma(\cdot)$ and $\beta(\cdot)$ for a sample of 10,000 stable vectors corresponding to $\Gamma_3$. LEFT column: Estimates for 80 gridpoints. RIGHT column: Estimates for 20 gridpoints. Vertical order of graphs and line types as in Figure 2.
sample estimates well enough.

We graph estimates of spectral measures in two formats. The first is a polar plot of the spectral measure that shows where the mass is located around the circle. The second is an $xy$ plot of the cumulative spectral measure as a function of the angle on the unit circle. The former has an intuitive appeal (showing independence and association) and works well when there are a few point masses. It does not work as well when there are many point masses and the “spikes” of the polar plot are noisy; perhaps smoothing this display would make it more useful.

The following examples use the spectral measures described in Section 2.

**Example 1.** In this example, the EDA ideas of Section 2 ruled out joint stability, so we do not show estimated spectral measures.

**Example 2.** We estimated $\Gamma_2$ (corresponding to independent components) on two evenly spaced grids of 100 and 24 points using the PROJ and ECF methods of estimation. For each of the grid sizes we plotted the estimated spectral measure on polar and cumulative plots. We also plotted the values of the $\sigma$ and $\beta$ for the sample projections and corresponding to both estimates of the spectral measure. Results of the above procedures are shown in Figure 4. We note that the large change in grid size did not affect the estimation very much. For both grid sizes the polar and cumulative plots show that the estimators put the correct amount of mass at roughly the correct place, with the total mass estimated accurately. The estimators of $\alpha$ were $\tilde{\alpha}_{ECF} = 1.198$ (same for both grid sizes), $\tilde{\alpha}_{PROJ} = 1.201$ with 100 gridpoints, and $\bar{\alpha}_{PROJ} = 1.199$ with 24 gridpoints. The graphs of $\sigma$ and $\beta$ show that the parameters corresponding to PROJ and ECF estimates of the measure are close to those estimated from the sample, and the effect of the grid size is minimal. The graph of $\sigma(\cdot)$ shows that the mass is spread around the unit circle and the graph of $\beta(\cdot)$ shows that the distribution is symmetric.

**Example 3.** We estimated $\Gamma_3$ on two evenly spaced grids of 80 and 20 points. The plots are presented in Figure 5, which again show that a large change in grid size does not affect the estimation very much. In this example, the plots of $\sigma(\cdot)$ and $\beta(\cdot)$ show that the distributions are heavily skewed: $\sigma(\cdot)$ has large swings and $\beta(\cdot)$ swings from $+1$ to $-1$ in different quadrants. The estimators of $\alpha$ were $\tilde{\alpha}_{ECF} = 1.510$ (same for both grid sizes), $\tilde{\alpha}_{PROJ} = 1.510$ with 80 gridpoints, and $\bar{\alpha}_{PROJ} = 1.509$ with 20 gridpoints.
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References


