A projection approach to multivariate stable laws

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Outline

- univariate stable laws & parameterizations
- univariate calculations & applications
- multivariate stable laws
- parameterizations & sub-classes
- computational formulas
- estimation
- metrics for comparing stable laws
- application: testing for independence

Univariate Stable Laws

Non-degenerate X is (sum or Lévy) stable iff for all $n > 1$, there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$X_1 + \cdots + X_n \stackrel{d}{=} c_n X + d_n,$$

where $X_1, \ldots, X_n$ are independent, identical copies of X. X is strictly stable iff $d_n = 0$ for all n.

Unimodal, heavy tailed, possibly skewed distributions with no closed formula for density or cdf (except for Gaussian, Cauchy, Lévy)

Four parameter family: index of stability $\alpha \in (0, 2]$, skewness $\beta \in [-1, 1]$,scale $\gamma > 0$,location δ. (α, β) determine the shape.

Characteristic Function

Standard parameterization:

$$E\exp(iuX) = \begin{cases} \exp\left(-\gamma^\alpha |u|^\alpha \left[1 - i\beta\,(\text{sign}\,u)\,\tan\frac{\pi\alpha}{2}\right] + i\delta_1 u\right) & \alpha \neq 1 \\ \exp\left(-\gamma |u| \left[1 + i\beta\,(\text{sign}\,u)\,\frac{2}{\pi}\log|u|\right] + i\delta_1 u\right) & \alpha = 1 \end{cases}$$

This is convenient for most theoretical purposes, but not for computational or some other purposes. It is not continuous in the neighborhood of $\alpha = 1$ and it is not a scale-location family when $\alpha = 1$.

Notation: $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta_1; 1)$.

Continuous parameterization

$$E \exp(iuX) = \exp\left(-\omega(\gamma u|\alpha,\beta;0) + i\delta_0 u\right)$$

$$\omega(u|\alpha,\beta;0) = \begin{cases} |u|^\alpha \left[1 + i\beta(\mathrm{sign}\, u) \tan\frac{\pi\alpha}{2}(|u|^{1-\alpha} - 1)\right] & \alpha \neq 1 \\ |u| \left[1 + i\beta(\mathrm{sign}\, u)\frac{2}{\pi} \log|u|\right] & \alpha = 1. \end{cases}$$

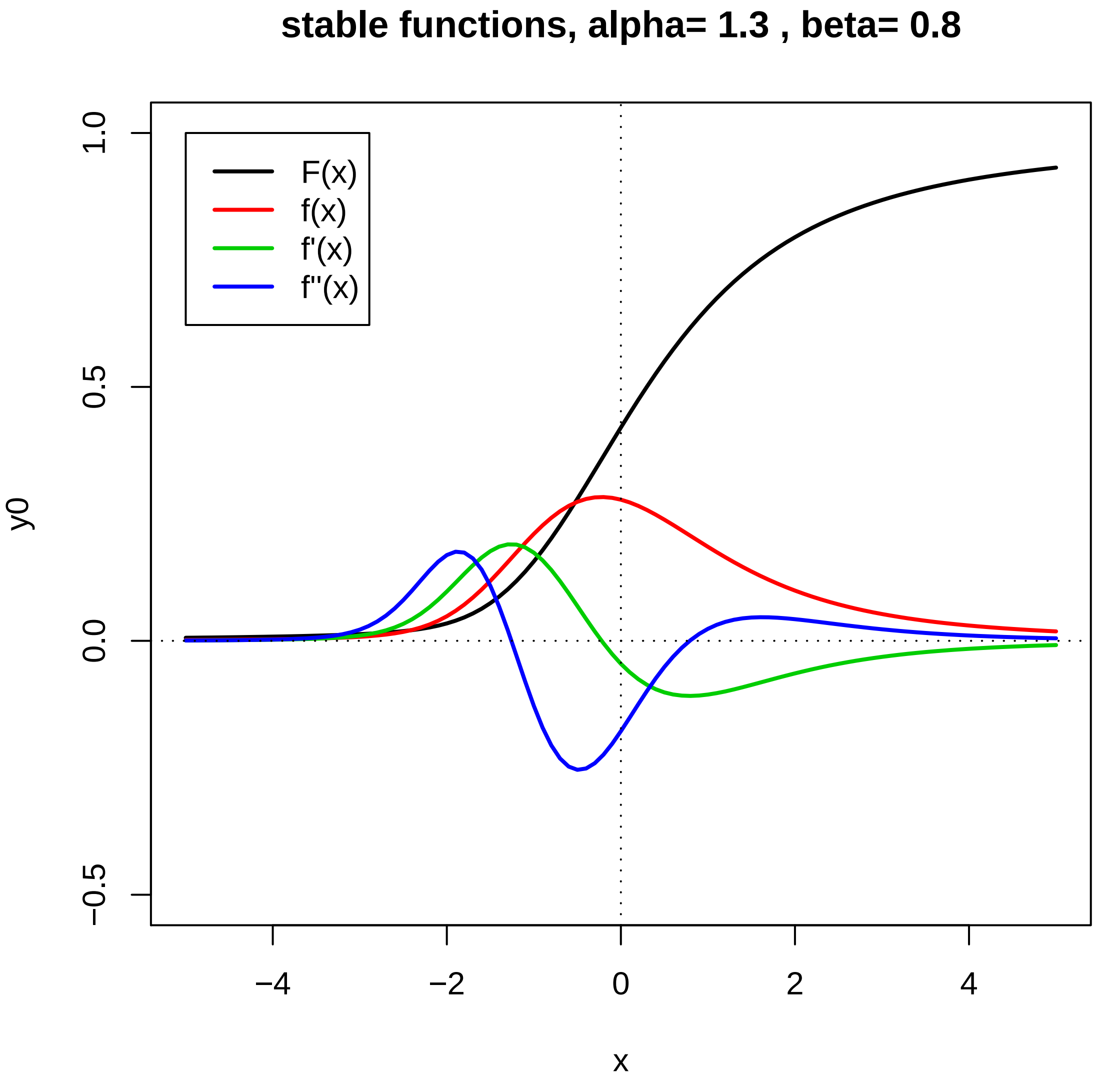
This is jointly continuous in all parameters and is a scale-location family for all α.

Notation: $X \sim \mathbf{S}(\alpha,\beta,\gamma,\delta_0;0)$.

Univariate computation

Effectively solved by STABLE program

- pdf f and cdf F by Zolotarev integrals
- fast approximations to pdf f and cdf F
- quantiles by numerically inverting F
- simulation by Chambers, Mallows & Stuck method
- estimation by max. likelihood, quantile, emp. char. fn., etc
- diagnostics for assessing stability

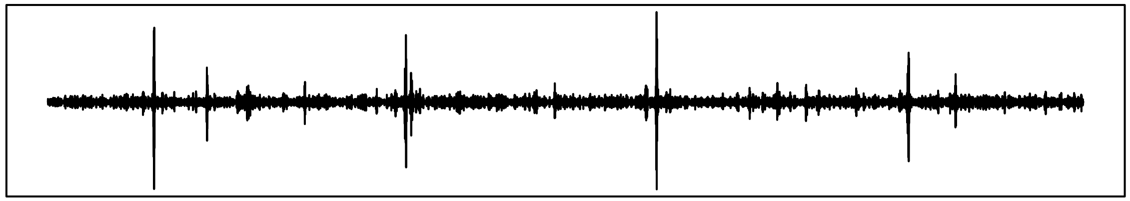

stable functions, alpha= 1.3 , beta= 0.8
F(x)
f(x)
f'(x)
f''(x)
y0
1.0
0.5
0.0
−0.5
−4
−2
0
2
4
x

Example: acoustic noise

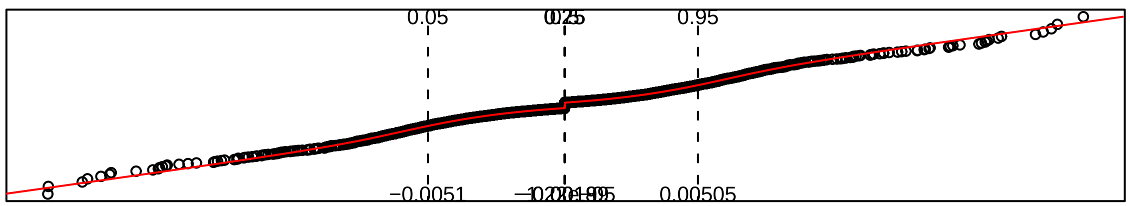
Alpheidae is a family of snapping shrimp

Data from Singapore, n =10,000 samples

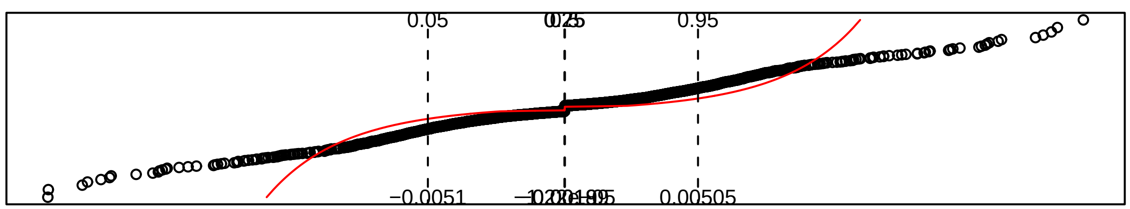
A21_BP_250kHz.txt



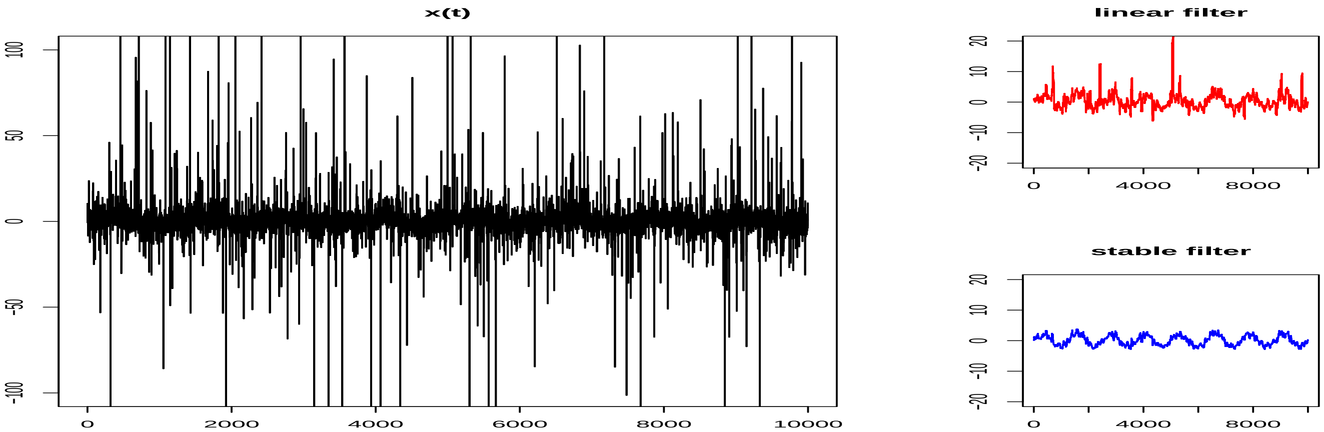
stable fit
theta=(1.7412,−0.0017,0.002,0)



normal fit
theta=(2,0,0.0031,0)



Sinusoidal example w/ stable noise: $\alpha = 1.3$, $\gamma = 2$, $n = 10000$, and window width $m = 50$.



Multivariate Stable Laws

Non-degenerate d-dim. r. vector $\mathbf{X}$ is stable if for any $n \geq 2$, there is a $c_n > 0$ and a vector $\mathbf{d}_n$ such that

$$\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n \stackrel{d}{=} c_n \mathbf{X} + \mathbf{d}_n,$$

where $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are i.i.d. copies of $\mathbf{X}$. $\mathbf{X}$ is strictly stable if $\mathbf{d}_n = 0$ for all n.

Projection characterization

Let $\mathbf{X}$ be a jointly stable random vector. Then for every $\mathbf{u} \in \mathbb{R}^d$, the one dimensional projection $\langle \mathbf{u}, \mathbf{X} \rangle = \sum u_i X_i$ is a univariate stable random variable with the same index α. (The converse is true if $\alpha \geq 1$; if $\alpha < 1$, an extra condition is needed.)

A stable random vector $\mathbf{X}$ is strictly stable if and only if all one dimensional projections $\langle \mathbf{u}, \mathbf{X} \rangle$ are strictly stable.

Projection parameterization

So, for $\mathbf{X}$ multivariate stable,

$$\langle \mathbf{u}, \mathbf{X} \rangle \sim \mathbf{S}\left(\alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}); k\right), \quad k = 0, 1.$$

By Cramer-Wold, these one dim. projections characterize the joint distribution. Explicitly, the joint characteristic function is

$$E \exp(i \langle \mathbf{u}, \mathbf{X} \rangle) = \phi(1 | \alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}); k),$$

where $\phi(\cdot | \alpha, \beta, \gamma, \delta; k)$ is the univariate char. function in k parameterization.

Notation: $\mathbf{X} \sim \mathbf{S}\left(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k\right)$ The functions $\beta(\cdot), \gamma(\cdot), \delta(\cdot)$ are called the projection parameter functions.

Projection parameterization II

Scaling properties of the projection parameter functions ($\gamma(r\mathbf{u}) = r\gamma(\mathbf{u})$, etc.) means it suffices to specify these functions on the unit sphere $\mathbb{S} \in \mathbb{R}^d$. These projection parameter functions capture the joint structure.

The symmetric case, centered at 0, is particularly simple:

$$E\,\text{exp}(i\langle\mathbf{u},\mathbf{X}\rangle) = \text{exp}\left(-\gamma^\alpha(\mathbf{u})\right) = \text{exp}\left(-|\mathbf{u}|^\alpha\gamma^\alpha(\mathbf{u}/|\mathbf{u}|)\right)$$

Spectral measure parameterization

Feldheim (1930s?): $\mathbf{X}$ stable, then there is a finite measure Λ on the unit sphere $\mathbb{S}$ and a vector δ with ($\omega(\cdot)$ on pg. 5):

$$E\,\exp(i\langle \mathbf{u}, \mathbf{X}\rangle) = \exp\left(-\int_{\mathbb{S}} \omega(\langle \mathbf{u}, \mathbf{s}\rangle)\,\Lambda(d\mathbf{s}) + i\langle \mathbf{u}, \delta\rangle\right).$$

Here the spread of mass by Λ determines the joint structure.

Notation: $\mathbf{X} \sim \mathbf{S}\,(\alpha, \Lambda, \delta; k)$.

Can express projection functions in terms of Λ, e.g. $\gamma^{\alpha}(\mathbf{u}) = \int_{\mathbb{S}} |\langle \mathbf{u}, \mathbf{s}\rangle|^{\alpha}\Lambda(d\mathbf{s})$, etc. Dependence of joint char. function on the spectral measure is solely in terms of these *parameter functions* $\beta(\cdot)$, $\gamma(\cdot)$, $\delta(\cdot)$.

No explicit way to recover Λ from the projection functions. (Can numerically get an approximation.)

Other parameterizations

Stochastic integrals: $X_i = \int_S f_i(s)M(ds)$ (ref. Samorodnisky and Taqqu). Here the dependence is captured in the integrand functions $f_1, \ldots, f_d$.

Zonoids ($\alpha \geq 1$) - associate a convex set B in $\mathbb{R}^d$ with $\mathbf{X}$ (ref. Molchanov). Here the dependence is captured in the shape of the region.

Advantage of the projection parameterization

Easy to state, builds on 1-dim. definition

Easy to handle linear transformations: if $\mathbf{X} \sim \mathbf{S}\,(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k)$ is d-dim. and A is a $m \times d$ matrix, $\mathbf{b} \in \mathbb{R}^m$, then $A\mathbf{X} + \mathbf{b}$ is m dim. α-stable with scale function $\gamma(A^T \cdot)$, skewness function $\beta(A^T \cdot)$, and location function $\delta(A^T \cdot) + \langle \cdot, \mathbf{b} \rangle$. (Contrast with spectral measure statement.)

Easy to handle sub-Gaussian case: $\gamma(\mathbf{u}) = (\mathbf{u}^T R \mathbf{u})^{1/2}$, where R is a positive definite shape matrix (essentially the covariance of the associated Gaussian vector).

Isotropic stable laws

Assume centered at 0, so $\mathbf{X} = A^{1/2}\mathbf{G}$, where A is positive $\alpha/2$-stable, $\mathbf{G}$ is N(0,I). In terms of the projection parameter functions:

$$\gamma(\mathbf{u}) = \gamma_0|\mathbf{u}|$$

In terms of spectral measure

$$\Lambda(d\mathbf{s}) = c\,d\mathbf{s} \quad \text{(surface measure)}$$

Occurs in radar signal processing and physics.

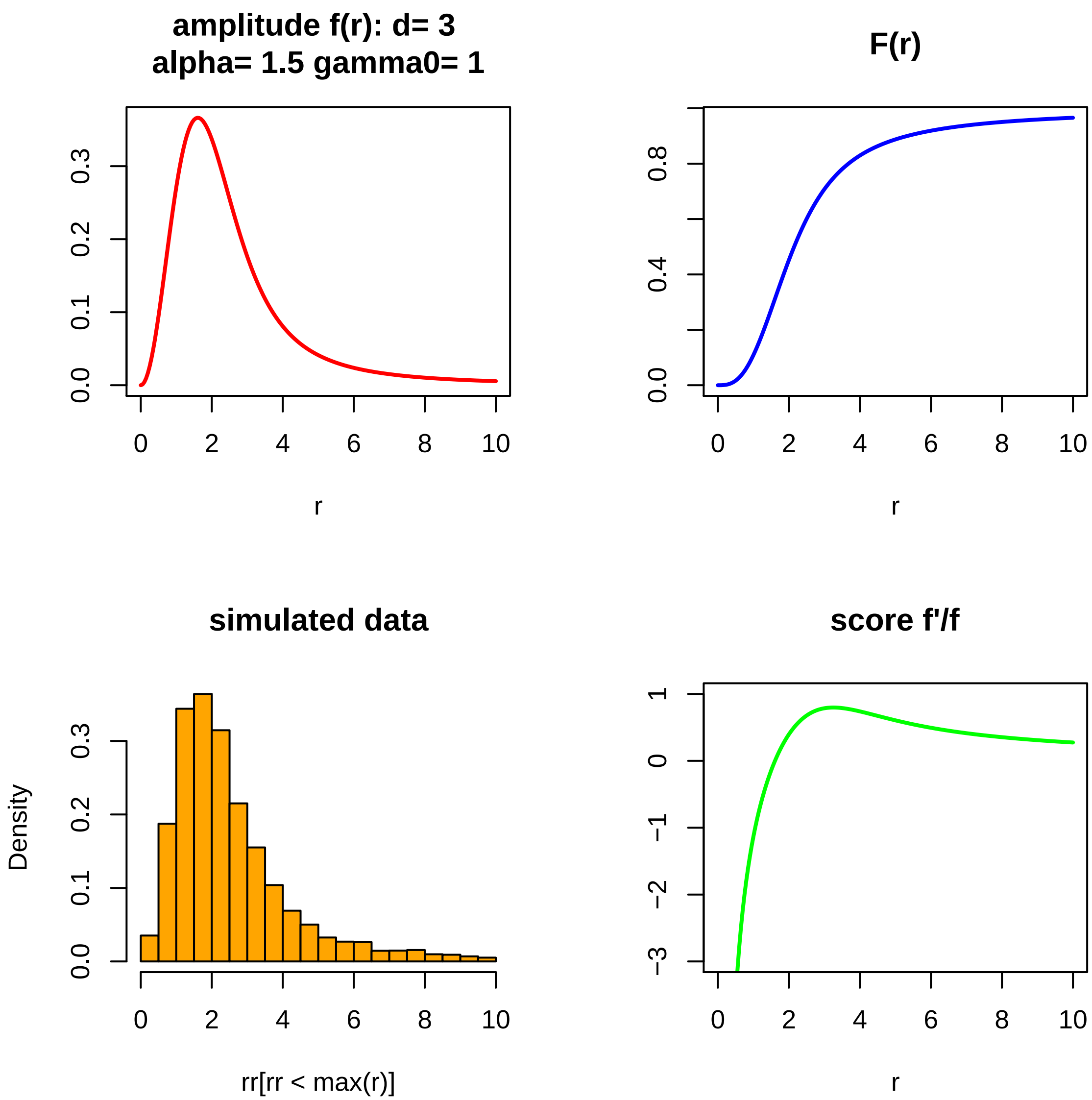
Radially symmetry reduces this to a one dimensional problem...

Isotropic: amplitude

Define the amplitude $R := |\mathbf{X}|$.

Since $R^2 = A(G_1^2 + \cdots + G_d^2) = AT$, where T is $\chi^2(d)$, you can evaluate the cdf $F_R(r)$ and pdf $f_R(r)$ numerically in terms of the cdf/pdf of A. With fast approximation to these, we can numerically evaluate this for dimensions $d \leq 100$.

Can also:
- find quantiles
- simulate via $\sqrt{AT}$
- given amplitude data, can estimate parameters (α, γ_0)
- compute score function $d/dr(\log f_R(r)) = f_R'(r)/f_R(r)$.



amplitude f(r): d= 3
alpha= 1.5 gamma0= 1
0.0
0.1
0.2
0.3
0
2
4
6
8
10
r
F(r)
0.0
0.4
0.8
0
2
4
6
8
10
r
simulated data
Density
0.0
0.1
0.2
0.3
0
2
4
6
8
10
rr[rr < max(r)]
score f'/f
−3
−2
−1
0
1
0
2
4
6
8
10
r

Isotropic: pdf and circular probabilities

d-dim. $\mathbf{X}$ isotropic α-stable:

$$P(|\mathbf{X}| \le r) = P(R \le r) = F_R(r \,|\alpha, \gamma_0, d)$$

The density of $\mathbf{X}$ is

$$f(\mathbf{x}) = \begin{cases} \Gamma(d/2) / \left(2\pi^{d/2}\right) |\mathbf{x}|^{1-d} f_R(|\mathbf{x}| \,|\alpha, \gamma_0, d) & \mathbf{x} \neq 0 \\ \Gamma(d/\alpha) / \left(\alpha 2^{d-1} \pi^{d/2} \Gamma(d/2)^2 \gamma_0^d\right) & \mathbf{x} = 0. \end{cases}$$

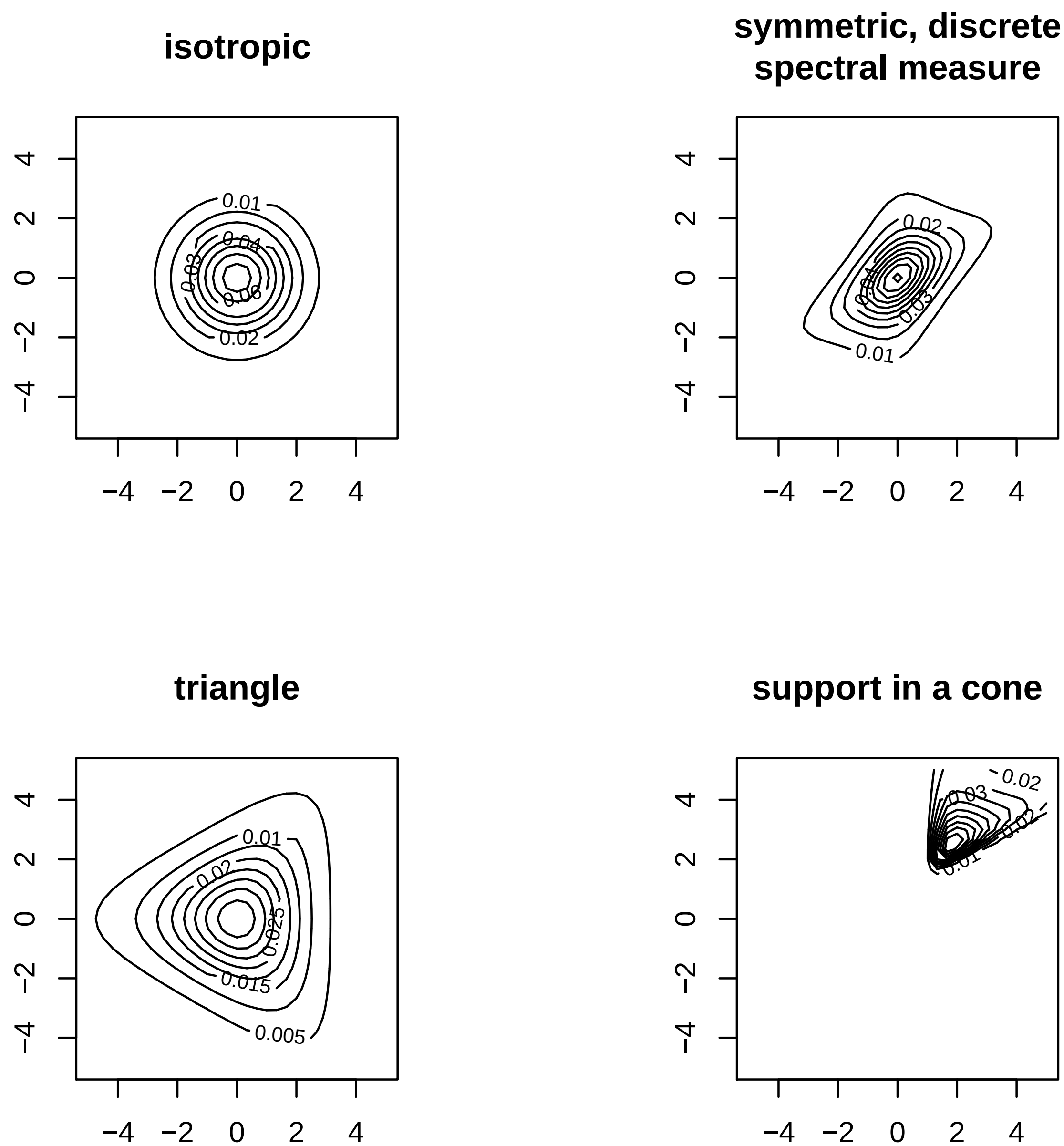
Other sub-classes

Independent components - can compute pdf, $P(\mathbf{X} \leq \mathbf{x})$, simulate.

Elliptically contoured - linear transformation of isotropic, so can compute pdf, simulate, find probability of being within a ellipse that is a level set.

Discrete spectral measure - integrals become sums. Can compute projection functions, simulate. This is a dense class (see below). Rajput and Nolan (1995) program to calculate two dim. densities by directly inverting ch. fn. Slow, limited accuracy and hard to implement when dim > 2.

General case - still hard to compute.


isotropic
0.01
0.04
0.03
0.06
0.02
symmetric, discrete
spectral measure
0.02
0.03
0.01
triangle
0.01
0.02
0.025
0.015
0.005
support in a cone
0.02
0.03
0.02
0.01
−4
−2
0
2
4

Density: general case

Define for $d > 0$

$$g_d(v|\alpha,\beta) = (2\pi)^{-d}\int_0^\infty \cos\left(vr + \beta\eta(r,\alpha)\right) r^{d-1} e^{-r^\alpha} dr$$

$\eta(r,\alpha) = \tan\frac{\pi\alpha}{2} r^\alpha$ $(\alpha \neq 1)$; $= (2/\pi) r \log r$ $(\alpha = 1)$. Abdul-Hamid, Nolan (1998) showed (1-parameterization)

$$f(\mathbf{x}) = \int_{\mathbb{S}} g_d\left(\left.\frac{\langle \mathbf{x},\mathbf{s}\rangle - \delta(\mathbf{s})}{\gamma(\mathbf{s})}\right| \alpha, \beta(\mathbf{s})\right) \gamma^{-d}(\mathbf{s}) d\mathbf{s}$$

Special functions $g_d(\cdot|\alpha,\beta)$ have intriguing properties, compute once and then can compute any d-dimensional stable density by integrating over compact set $\mathbb{S}$.

Existing program to calculate $g_2(\cdot|\alpha, 0)$, so can compute $f(\mathbf{x})$ for 2-dim. symmetric case (at least for $|\mathbf{x}|$ moderate).

Probability: general case

Can calculate $P(\mathbf{X} \in [\mathbf{a}, \mathbf{b}])$ in two dim. when symmetric by numerically integrating $f(\mathbf{x})$ or in non-symmetric case or higher dimensions by Monte Carlo estimation.

New approach: define

$$
\begin{aligned}
g_0(x|\alpha,\beta) &= \int_0^\infty \frac{\cos(xr+\beta\eta(r,\alpha)) - \cos(\beta\eta(r,\alpha))}{r} e^{-r^\alpha} dr \\
\tilde{g}_0(x|\alpha,\beta) &= \int_0^\infty \frac{\sin(xr+\beta\eta(r,\alpha)) - \sin(\beta\eta(r,\alpha))}{r} e^{-r^\alpha} dr \\
\tilde{g}_d(v|\alpha,\beta) &= (2\pi)^{-d} \int_0^\infty \sin\left(vr+\beta\eta(r,\alpha)\right) r^{d-1} e^{-r^\alpha} dr, \quad d>0
\end{aligned}
$$

Also define

$$G_d(x|\alpha,\beta) = \begin{cases} \sum_{j=0}^{m-1} c_{d,2j} x^{2j} g_{2j}(x|\alpha,\beta) + \sum_{j=0}^{m-1} \tilde{c}_{d,2j+1} x^{2j+1} \tilde{g}_{2j+1}(x|\alpha,\beta) \\ \qquad \text{d=2m} \\ \sum_{j=0}^{m-1} c_{d,2j+1} x^{2j+1} g_{2j+1}(x|\alpha,\beta) + \sum_{j=0}^{m} \tilde{c}_{d,2j} x^{2j} \tilde{g}_{2j}(x|\alpha,\beta) \\ \qquad \text{d=2m+1} \end{cases}$$

$c_{d,k}$ and $\tilde{c}_{d,k}$ are constants depending on α only.

Probability II

In the univariate case, the standardized d.f. is given by

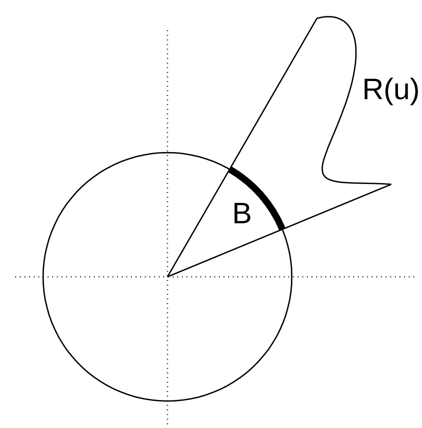
$$F(x|\alpha,\beta) - F(0|\alpha,\beta) = \int_0^x f(x|\alpha,\beta) = \frac{1}{\pi}\widetilde{g}_0(x|\alpha,\beta), \quad x > 0.$$

We note that there are explicit formulas for $F(0|\alpha,\beta)$ when $\alpha \neq 1$ and that the d.f. formula is naturally in "polar form", giving the probability of being in the interval $(0, x]$.

Assume the Borel set $A \subset \mathbb{R}^d$ has a polar decomposition, i.e. $A \leftrightarrow (B, R(\cdot))$ where $B \subset \mathbb{S}$ and $R(\mathbf{u})$ is a nonnegative function defined on B:

$$A = \{r\mathbf{u} : \mathbf{u} \in B, 0 \leq r < R(\mathbf{u})\}$$

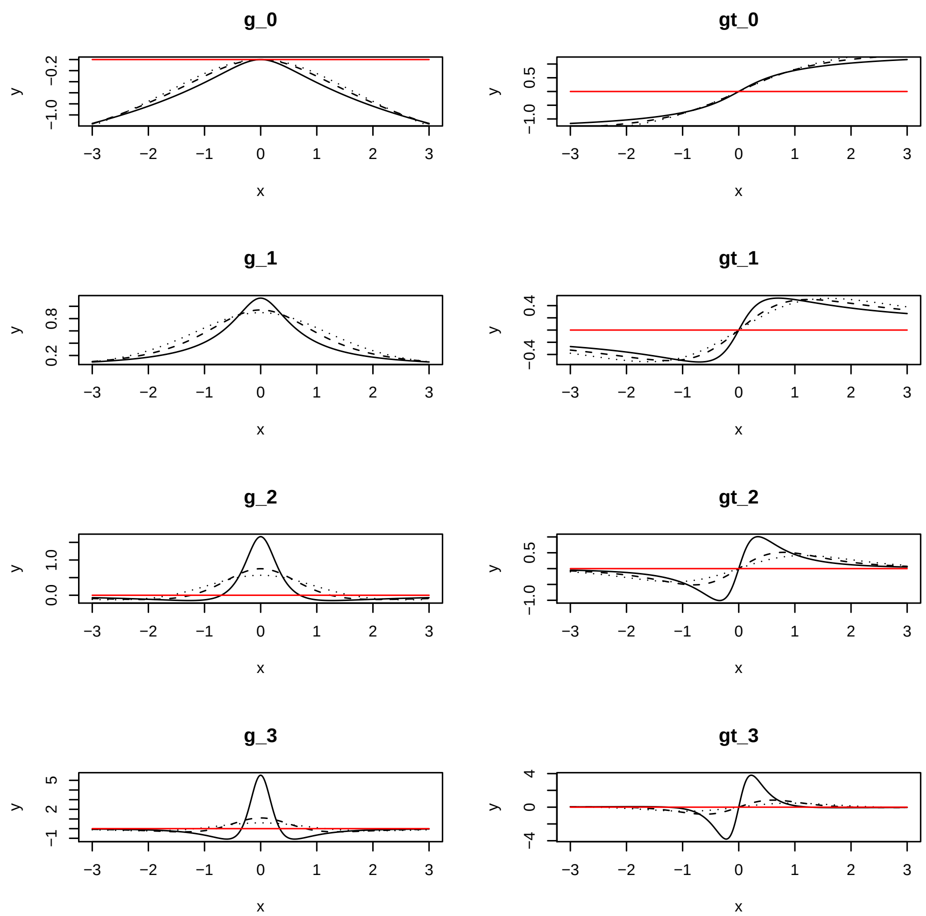
A is the cone generated by base B and bounded by $R(\cdot)$.

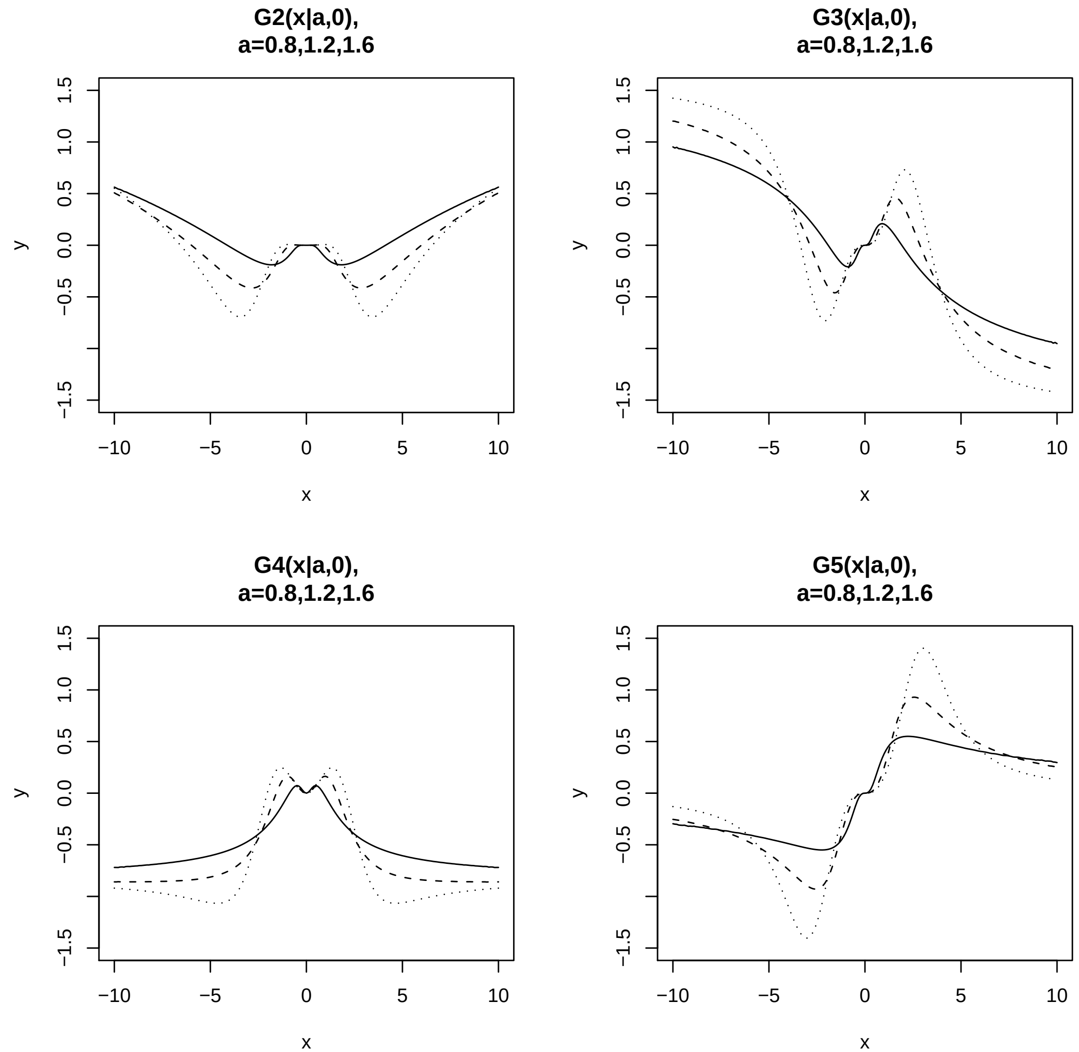


Let $\mathbf{X} \sim \mathbf{S}\,(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 1)$ be a nonsingular strictly stable d-dimensional random vector and let A be a bounded set expressed in polar form as $(B, R(\cdot))$. When $\alpha \neq 1$

$$P(\mathbf{X} - \delta \in A) = (2\pi)^{-d} \int_B \int_{\mathbb{S}} G_d \left(\left. \frac{R(\mathbf{u})\langle \mathbf{u}, \mathbf{s}\rangle}{\gamma(\mathbf{s})} \right| \alpha, \beta(\mathbf{s}) \right) \langle \mathbf{u}, \mathbf{s}\rangle^{-d} \, d\mathbf{s} \, d\mathbf{u}.$$

Similar formula when $\alpha = 1$.


g_0
gt_0
g_1
gt_1
g_2
gt_2
g_3
gt_3
x
y


G2(x|a,0),
a=0.8,1.2,1.6
G3(x|a,0),
a=0.8,1.2,1.6
G4(x|a,0),
a=0.8,1.2,1.6
G5(x|a,0),
a=0.8,1.2,1.6
y
x
−1.5
−0.5
0.0
0.5
1.0
1.5
−10
−5
0
5
10

Stable integrals

To access asymmetric and $d > 2$ cases, we can use Zolotarev type integrals. Start with

$$\mathbf{S}(x|\alpha, \beta) = \int_0^\infty e^{-irx}\phi(r)dr$$

This function and it's derivatives and integral yield expressions for multivariate stable densities and cumulative probabilities. First, a Zolotarev type integral for $\mathbf{S}$.

Define

$$\theta_0 = \begin{cases} \alpha^{-1} \arctan\left(\beta \tan \frac{\pi\alpha}{2}\right) & \alpha \neq 1 \\ \pi/2 & \alpha = 1 \end{cases}$$

$$y_0 = \begin{cases} (\cos(\alpha\theta_0)x/\alpha)^{1/(\alpha-1)} & (\alpha > 1, \beta = -1) \text{ or} \\ & (\alpha < 1, \beta = 1) \text{ and } x > 0 \\ +\infty & \alpha < 1, \beta = -1, x > 0 \\ \exp(-\pi x/2 - 1) & \alpha = 1, x \in \mathbb{R}. \end{cases}$$

$$V(\theta|\alpha,\beta) = \begin{cases} (\cos\alpha\theta_0)^{\frac{1}{\alpha-1}} \left(\dfrac{\cos\theta}{\sin\alpha(\theta_0+\theta)}\right)^{\frac{\alpha}{\alpha-1}} \dfrac{\cos(\alpha\theta_0 + (\alpha-1)\theta)}{\cos\theta} & \alpha \neq \\ \frac{2}{\pi}\left(\dfrac{\frac{\pi}{2}+\beta\theta}{\cos\theta}\right) \exp\left(\dfrac{1}{\beta}\left(\frac{\pi}{2}+\beta\theta\right)\tan\theta\right) & \alpha = \end{cases}$$

$$\widetilde{V}(\theta|\alpha,\beta) = \begin{cases} -\tan\theta + \dfrac{\alpha-1}{\alpha}\dfrac{\sin\alpha(\theta_0+\theta)}{\cos\theta\cos(\alpha\theta_0+(\alpha-1)\theta)} & \alpha \neq 1 \\ -\tan\theta - \dfrac{\beta}{\pi/2+\beta\theta} & \alpha = 1, \beta > 0. \end{cases}$$

$V(\theta) > 0$ and $\widetilde{V}(\theta)$ are both monotonic.

When $\alpha \in (0,1) \cup (1,2)$ and $\beta \in [-1,1]$, for any $x > 0$

$$\mathbf{S}(x) = \frac{\alpha x^{\frac{1}{\alpha-1}}}{|\alpha - 1|} \int_{-\theta_0}^{\pi/2} \left[1 + i\widetilde{V}(\theta)\right] V(\theta) e^{-x^{\frac{\alpha}{\alpha-1}} V(\theta)} d\theta + i\widetilde{W}(x).$$

When $\alpha = 1$, $\beta \in (0,1]$, for any $x \in \mathbb{R}$

$$\mathbf{S}(x) = \frac{\pi}{2\beta} e^{-\frac{\pi x}{2\beta}} \int_{-\pi/2}^{\pi/2} \left[1 + i\widetilde{V}(\theta)\right] V(\theta) e^{-e^{-\frac{\pi x}{2\beta}} V(\theta)} d\theta + i\widetilde{W}(x).$$

Set $\epsilon = \text{sign}\,(\alpha - 1)$,

$$\widetilde{W}(x) = \begin{cases} -\beta \int_0^{y_0} \exp\left(\beta x y + \epsilon y^{\alpha}/k_0\right) dy & (\alpha < 1, \beta = \pm 1) \text{ or} \\ & (\alpha > 1, \beta = -1), x > 0 \\ \int_0^{y_0} \exp\left(xy + (2/\pi) y \log y\right) dy & \alpha = 1, \beta = 1, x \in \mathbb{R} \\ 0 & \text{otherwise.} \end{cases}$$

s_d functions

Define

$$\begin{aligned}\mathbf{S}_0(x|\alpha,\beta) &= \int_0^x \mathbf{S}(t|\alpha,\beta)dt \\ \mathbf{S}_1(x|\alpha,\beta) &= \mathbf{S}(x|\alpha,\beta) \\ \mathbf{S}_d(x|\alpha,\beta) &= \frac{\partial^{d-1}\mathbf{S}(x|\alpha,\beta)}{\partial x^{d-1}}\end{aligned}$$

Then

$$\mathbf{S}_d(x|\alpha,\beta) = (-i)^{d-1}\left[g_d(x|\alpha,\beta) - i\tilde{g}_d(x|\alpha,\beta)\right], \quad d = 0,1,2,\ldots$$

Expressions for $\mathbf{S}_d(0|\alpha,\beta)$ and $\mathbf{S}_d(x|1,0)$. Can get Zolotarev type integrals and recursive relations for g_d and $\tilde{g}_d$. In particular, get computational formulas for $f'(x)$ and $f''(x)$.

Multivariate Estimation

Estimate α and a discrete estimate of the spectral measure $\hat{\Lambda} = \sum_{k=1}^{m} \lambda_k \delta_{\mathrm{s}_k}$ by 3 methods:
- tail/empirical processes (Rachev-Xin-Cheng)
- sample char. function (Nolan, Panorska)
- projection method (Nolan, Panorska, McCulloch): estimate projection parameter functions

Works ok in 2 dim., difficult in higher dimensions.

Separate methods for isotropic or elliptical cases, works for higher dimensions $d \leq 100$.

Mixtures of extreme value distributions

Fougères, Nolan, Rootzen (2008)

For a vector $\mathbf{y} = (y_1, \ldots, y_d)$, the p^{th} power is defined to be the vector $\mathbf{y}^p := (y_1^p, \ldots, y_d^p)$. Let $X_1, \ldots, X_d$ be i.i.d. univariate standardized ($\sigma = 1, \mu = 0$) Frechét with shape ξ. Let $\mathbf{S} = (S_1, \ldots, S_d)$ be a centered positive α-stable random vector with scale function $\gamma(\cdot)$ and independent of $X_1, \ldots, X_d$. Then

$$\mathbf{Y} = \left(X_1 S_1^{1/\xi}, \ldots, X_d S_d^{1/\xi} \right)$$

is a multivariate Frechét distribution with shape parameter $\alpha\xi$ and d.f.

$$P(\mathbf{Y} \le \mathbf{y}) = \exp\left(-c_\alpha \, \gamma^\alpha (y^{-\xi}) \right)$$

Metrics - Background

Byczkowski, Nolan and Rajput (1993) - approximating general spectral measures with discrete spectral measures

Davydov and Paulauskas (1999) - closeness of two symmetric stable distributions for general spectral measures

Here: generalize to nonsymmetric case, use continuous parameterization so that results are uniform in α, use both projection parameter functions; then relate to spectral measure.

We will always use 0-parameterization to get results uniform in α.

For $\mathbf{X}_i \sim S(\alpha_i, \beta_i(\cdot), \gamma_i(\cdot), \delta_i(\cdot); 0)$, $i = 1, 2$ define for $p \in [1, \infty]$:

$\Delta_p(\mathbf{X}_1, \mathbf{X}_2) = |\alpha_1 - \alpha_2| + \|\beta_1(\cdot) - \beta_2(\cdot)\|_p + \|\gamma_1(\cdot) - \gamma_2(\cdot)\|_p + \|\delta_1(\cdot) - \delta_2(\cdot)\|_p$,

($L^p(\mathbb{S}, d\mathbf{s})$ norm, $d\mathbf{s}$ is surface area measure on $\mathbb{S}$, NOT on $\mathbb{R}^d$).

Let $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ be α_j-stable r. vectors.

$\mathbf{X}_j \xrightarrow{d} \mathbf{X}_0$ as $j \to \infty$ if and only if $\Delta_\infty(\mathbf{X}_j, \mathbf{X}_0) \to 0$.

Main result

$\gamma_{\text{min}} = \inf_{\mathbf{u}\in\mathbb{S}} \gamma(\mathbf{u}) > 0$ iff $\mathbf{X}$ has a density $f(\mathbf{x})$ on $\mathbb{R}^d$.

Consider two distributions: $\mathbf{X}_1$ and $\mathbf{X}_2$, respective densities $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ exist iff

$$\underline{\gamma} := \min(\gamma_{\text{min}}(\mathbf{X}_1), \gamma_{\text{min}}(\mathbf{X}_2)) > 0.$$

For $\mathbf{X}_i \sim S(\alpha_i, \beta_i(\cdot), \gamma_i(\cdot), \delta_i(\cdot); 0)$, $i = 1, 2$ any stable r. vectors with $\underline{\gamma} > 0$:

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \leq c(\alpha_1, \alpha_2, d, \underline{\gamma})\Delta_1(\mathbf{X}_1, \mathbf{X}_2)$$

For non-probability measures, define the extended Prohkorov metric π^*: $\lambda_j := \Lambda_j(\mathbb{S})$, $\pi =$ Prohkorov metric,

$$\pi^*(\Lambda_1, \Lambda_2) = |\lambda_1 - \lambda_2| + \min(\lambda_1, \lambda_2)\pi\left(\frac{\Lambda_1(\cdot)}{\lambda_1}, \frac{\Lambda_2(\cdot)}{\lambda_2}\right)$$

Then for $\mathbf{X}_j \sim \mathbf{S}\left(\alpha_j, \Lambda_j, \delta_j; 0\right)$, $j = 1, 2$, $\underline{\gamma} > 0$:

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \leq c(\alpha_1, \alpha_2, d, \underline{\gamma})\left[|\alpha_1 - \alpha_2| + \pi^*(\Lambda_1, \Lambda_2)^{\max(\alpha_1,\alpha_2)/2}\right]$$

Davydov and Paulauskas (1999) - symmetric case, directly use char. function. Our proof uses above parameter function metric and the following.

Main idea of the proof is that if $\underline{\gamma} > 0$, then

$$\begin{aligned}
\|\beta_1(\cdot) - \beta_2(\cdot)\|_\infty &\leq c_1(\alpha, \underline{\gamma}, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\min(1,\alpha)} \\
\|\gamma_1(\cdot) - \gamma_2(\cdot)\|_\infty &\leq c_2(\alpha, \underline{\gamma}, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\min(1,\alpha)} \\
\|\delta_1(\cdot; 0) - \delta_2(\cdot; 0)\|_\infty &\leq |\delta_1 - \delta_2| + c_3(\alpha, \underline{\gamma}, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\alpha/2}.
\end{aligned}$$

Idea: spectral measures close in Prohkorov sense $\Rightarrow$
parameter functions close pointwise and in L^1 $\Rightarrow$
distributions close.

Results show that finite discrete spectral measures are dense. Discrete spectral measures are easy to work with for simulation, density calculations, estimation.

Can get similar results on cumulative probabilities, showing a uniform bound for all Borel sets A:

$$|P(\mathbf{X}_1 \in A) - P(\mathbf{X}_2 \in A)| \le c(\alpha_1, \alpha_2, d, \underline{\gamma})\, \mathbf{\Delta}_1(\mathbf{X}_1, \mathbf{X}_2)$$

$\mathbf{\Delta}_1(\mathbf{X}_1, \mathbf{X}_2)$ is easy to approximate using any one-dimensional estimation method (max. likelihood, quantile, empirical ch. function, etc), $\pi^*(\Lambda_1, \Lambda_2)$ is not.