Multivariate elliptically contoured stable distributions: theory and estimation

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Abstract

Mulitvariate stable distributions with elliptical contours are a class of heavy tailed distributions that can be useful for modeling financial data. This paper describes the theory of such distributions, presents formulas for calculating their densities, methods for fitting the data and assessing the fit. Numerical routines are described that work for dimension $d \leq 40$. An example looks at a portfolio with 30 assets.

1 Introduction

Stable distributions are a class of probability distributions that generalize the normal law, allowing heavy tails and skewness that make them attractive in modeling financial data. While there are many attractive theoretical properties of stable laws, the use of these models in practice has been restricted by the lack of formulas for stable densities and distribution functions. The univariate stable distributions are now mostly accessible. There are reliable programs to compute stable densities, distribution functions, and quantiles. And there are fast methods to simulate stable r.v.s and several methods of estimating stable parameters based on maximum likelihood, quantiles, empirical characteristic functions, and fractional moments, see Nolan (2001).

On the other hand, multivariate stable laws are only partially accessible. This is a function of the lack of closed form expressions for densities, and the possible complexity of the dependence structures. Byczkowski et al. (1993), Abdul-Hamid and Nolan (1998) and Nolan (2007) give expressions for general multivariate stable densities and distribution functions. In the bivariate case, there are some methods of computing densities and estimating, but these are difficult to implement in higher dimensions. This paper focuses on a computationally tractable case - elliptically contoured stable laws. It is shown that one can compute densities, approximate cumulative probabilities and fit elliptical stable distributions in dimension $d \leq 40$.

If $\mathbf{X}$ is α-stable and elliptically contoured, then it has joint characteristic function

$$E \exp(i\mathbf{u}^T\mathbf{X}) = \exp(-(\mathbf{u}^T\Sigma\mathbf{u})^{\alpha/2} + i\mathbf{u}^T\boldsymbol{\delta}) \tag{1}$$

for some positive definite matrix Σ and shift vector $\boldsymbol{\delta} \in \mathbb{R}^d$. Here $\mathbf{x}^T\mathbf{y} = \sum_{k=1}^d x_i y_i$ is the inner product in $\mathbb{R}^d$. The spectral measure of this stable laws is known, but complicated; see Proposition 2.5.8 of Samorodnitsky and Taqqu (1994). We will call the matrix Σ the *shape matrix* of the elliptical distribution.

We assume throughout that $\mathbf{X}$ is nonsingular, which is equivalent to Σ being strictly positive definite, i.e. for every $\mathbf{u} \neq 0$, $\mathbf{u}^T\Sigma\mathbf{u} > 0$. All elliptically contoured stable distributions are scale mixtures of multivariate normal distributions, see Proposition 2.5.2 of Samorodnitsky and Taqqu (1994). Let $\mathbf{G} \sim N(\mathbf{0}, \Sigma)$ be a d-dimensional multivariate normal r. vector and $A \sim \mathbf{S}(\alpha/2, 1, \gamma, 0)$ be an independent univariate positive $(\alpha/2)$-stable r. v. with $0 < \alpha < 2$. Then $\mathbf{X} = A^{1/2}\mathbf{G}$ is α-stable elliptically contoured with joint characteristic function

$$\exp(-(\gamma/2)^{\alpha/2}(\sec \pi\alpha/4)(\mathbf{u}^T\Sigma\mathbf{u})^{\alpha/2}).$$

For this reason, elliptically contoured stable distributions are called sub-Gaussian stable. This gives a formula for simulating elliptical stable distributions. In particular, if $0 < \alpha < 2$, $A \sim \mathbf{S}(\alpha/2, 1, 2\gamma_0^2(\cos \pi\alpha/4)^{2/\alpha}, 0)$ and $\mathbf{G} \sim N(0, \Sigma)$, then

$$\mathbf{X} = A^{1/2}\mathbf{G} + \boldsymbol{\delta}$$

has characteristic function (1).

The isotropic/radially symmetric cases arise when Σ is a multiple of the identity matrix, in which case the characteristic function simplifies to

$$E \exp(i\mathbf{u}^T\mathbf{X}) = \exp(-\gamma_0^\alpha|\mathbf{u}|^\alpha + i\mathbf{u}^T\boldsymbol{\delta}) \tag{2}$$

where $\gamma_0 > 0$ is a scale parameter and $\boldsymbol{\delta} \in \mathbb{R}^d$ is a location parameter. The spectral measure in this case is a uniform distribution on the unit sphere $\mathbb{S} = \{\mathbf{x}^T\mathbf{x} = 1\} \subset \mathbb{R}^d$. If A is as above and $\mathbf{G} \sim N(\mathbf{0}, I)$, then $\mathbf{X} = A^{1/2}\mathbf{G} + \boldsymbol{\delta}$ has characteristic function (2).

The organization of this paper is as follows. Section 2 focuses on a special case: the radially symmetric or isotropic case. Here the radial symmetry allows one to characterize the joint distribution in terms of the amplitude $R = |\mathbf{X}|$. This univariate random variable can be numerically evaluated, and provides a way of evaluating the multivariate isotropic stable densities. Section 3 treats the elliptically contoured stable laws, shows how to compute these multivariate densities, and discusses estimation of this model. We end with an application, where the 30 stocks in the Dow Jones index are jointly analyzed as an elliptical stable model with $\alpha = 1.71$. An appendix gives more facts about the amplitude distribution.

2 Isotropic stable distributions

2.1 The amplitude distribution

Let $\mathbf{X}$ be a centered d-dimensional isotropic stable random vector with characteristic function $\exp(-\gamma_0^\alpha |\mathbf{u}|^\alpha)$. The *amplitude* of $\mathbf{X}$ is defined by

$$R = |\mathbf{X}| = \sqrt{X_1^2 + \cdots + X_d^2}.$$

Our primary interest here is in using the distribution of univariate R to get expressions for the density of multivariate isotropic and elliptical stable distributions. However, in some problems the amplitude arises directly, so it is worthwhile exploring it's properties. This section derives expressions for it's density and d.f. for general dimension. In dimension $d = 1$, isotropic is equivalent to symmetric, and the cumulative distribution function of $R = |X|$ is $F_R(r) = P(|X| \le r) = F_X(r) - F_X(-r) = 2F_X(r) - 1$ and the density is $f_R(r) = 2f_X(r)$. For the rest of this paper we assume $d \ge 2$.

When $0 < \alpha < 2$, $\mathbf{X} \overset{d}{=} A^{1/2}\mathbf{Z}$, where $A \sim \mathbf{S}(\alpha/2, 1, 2\gamma_0^2(\cos \pi\alpha/4)^{2/\alpha}, 0)$ is positive stable and $\mathbf{Z} \sim N(\mathbf{0}, I)$, A and $\mathbf{Z}$ independent. Thus

$$R^2 \overset{d}{=} A(Z_1^2 + \cdots + Z_d^2) = AT, \tag{3}$$

where T is chi-squared with d degrees of freedom, and independent of A. Using the standard formula for products of independent r.v., the d.f. of R can be expressed as

$$F_R(r) = F_R(r|\alpha, \gamma_0, d) = P(R \le r) = P(AT \le r^2) = \int_0^\infty F_A(r^2/t) f_T(t) dt, \tag{4}$$

and the density as

$$f_R(r) = f_R(r|\alpha, \gamma_0, d) = F_R'(r) = 2r \int_0^\infty f_A(r^2/t) \frac{f_T(t)}{t} dt. \tag{5}$$

A scaling argument shows $F_R(r|\alpha, \gamma_0, d) = F_R(r/\gamma_0|\alpha, 1, d)$ and $f_R(r|\alpha, \gamma_0, d) = f_R(r/\gamma_0|\alpha, 1, d)/\gamma_0$. Figure 1 shows the graph of the density in two and three dimensions.

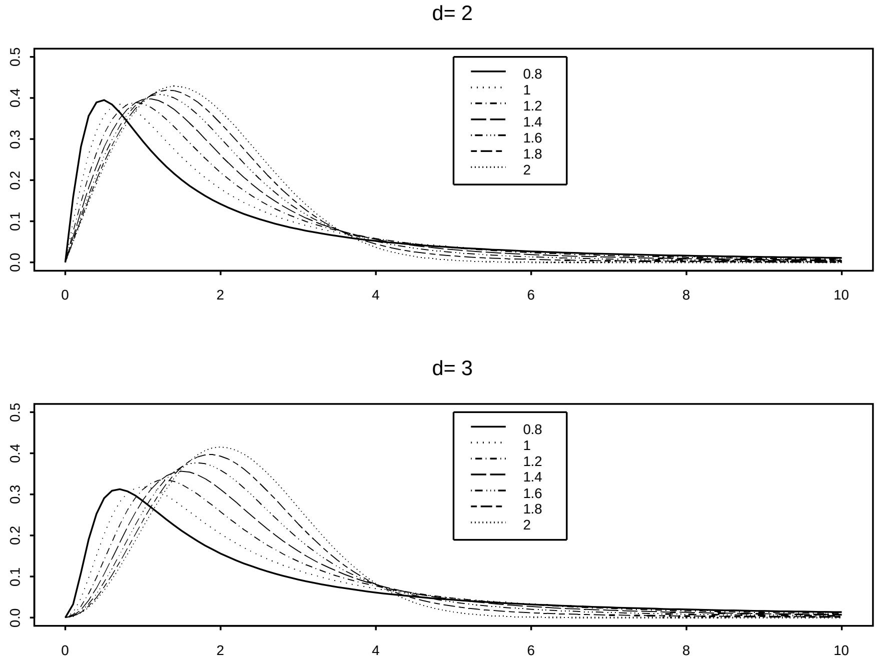

Figure 1: The density of the standardized ($\gamma_0 = 1$) amplitude in 2 dimensions (top) and 3 dimensions (bottom) for $\alpha = 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2$.

Equation (3) gives a way of simulating the amplitude distribution directly, without having to generate multivariate $\mathbf{X}$. It also gives an alternative way of simulating radially symmetric stable random vectors in d dimensions: let $A \sim \mathbf{S}(\alpha/2, 1, 2\gamma_0^2(\cos\pi\alpha/4)^{2/\alpha}, 0)$, $T \sim \chi^2(d)$, and $\mathbf{S}$ uniform on $\mathbb{S}$, then $\mathbf{X} \overset{d}{=} \sqrt{AT}\mathbf{S}$ is radially symmetric α-stable with scale γ_0. In particular, in two dimensions, T is exponential and can be generated by $-2\log U_1$ and $\mathbf{S} = (\cos(2\pi U_2), \sin(2\pi U_2))$, where U_1 and U_2 are independent U(0,1).

There are other expressions for the amplitude distribution. One is a simple change of variables: setting $s = r^2/t$ transforms (4) and (5) to

$$F_R(r) = r^2 \int_0^\infty s^{-2} F_A(s) f_T(r^2/s) ds = \frac{r^d}{2^{d/2}\Gamma(d/2)} \int_0^\infty F_A(s) s^{-d/2-1} e^{-r^2/(2s)} ds \quad (6)$$

$$f_R(r) = 2r\int_0^\infty s^{-1} f_A(s) f_T(r^2/s)ds = \frac{2r^{d-1}}{2^{d/2}\Gamma(d/2)}\int_0^\infty f_A(s) s^{-d/2} e^{-r^2/(2s)} ds \tag{7}$$

A third expression is from Zolotarev (1981):

$$f_R(r) = \frac{2}{2^{d/2}\Gamma(d/2)}\int_0^\infty (rt)^{d/2} J_{d/2-1}(r\,t) e^{-\gamma_0^\alpha\, t^\alpha} dt, \tag{8}$$

where $J_\nu(\cdot)$ is the Bessel function of order ν.

The program to compute f_R and F_R evaluates (4) and (5) by numerical integration using existing routines to calculate the univariate stable d.f. F_A or the univariate stable density f_A. The current program works for $\alpha \geq 0.8$ and dimensions $1 \leq d \leq 40$. The integral in (8) is more difficult to evaluate numerically, because the integrand oscillates infinitely many times, whereas the integrands in (4) and (5) do not.

More facts about the amplitude density and d.f. are given in the Appendix. The series expansions for the amplitude d.f. and density from there show:

$$\lim_{r\to\infty} r^\alpha(1 - F_R(r)) = \lim_{|\mathbf{X}|\to\infty} r^\alpha P(R > r) = k_1\gamma_0^\alpha \tag{9}$$

$$\lim_{r\to 0} r^{-d} F_R(r) = k_2\gamma_0^{-d} \tag{10}$$

$$\lim_{r\to\infty} r^{\alpha+1} f_R(r) = \alpha k_1\gamma_0^\alpha \tag{11}$$

$$\lim_{r\to 0} r^{1-d} f_R(r) = d k_2\gamma_0^{-d} \tag{12}$$

for positive constants

$$k_1 = k_1(\alpha, d) = 2^\alpha \frac{\sin(\pi\alpha/2)}{\pi\alpha/2}\frac{\Gamma((\alpha+2)/2)\Gamma((\alpha+d)/2)}{\Gamma(d/2)},$$

$$k_2 = k_2(\alpha, d) = \frac{4\Gamma(d/\alpha)}{\alpha 2^d \Gamma(d/2)^2}.$$

We note that R is not stable, but (9) shows R is in the domain of attraction of a univariate α-stable law with $\beta = 1$.

2.2 Densities of isotropic stable distributions

Let $\mathbf{X}$ be any radially symmetric (around $\mathbf{0}$) r. vector, not necessarily stable, with density $f_{\mathbf{X}}(x)$ and amplitude $R = |\mathbf{X}|$. The d.f. of R, $F_R(r) = P(|\mathbf{X}| \leq r)$, directly gives circular probabilities. The following argument gives an expression for the density of X in terms of the density of R. Using polar coordinates and radially symmetry for $r > 0$,

$$\begin{aligned} F_R(r) &= P(|\mathbf{X}| \le r) = \int_{|\mathbf{x}|\le r} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} = \int_0^r \int_{\mathbb{S}} f_{\mathbf{X}}(u\mathbf{s})u^{d-1}d\mathbf{s}du \\ &= \int_0^r \int_{\mathbb{S}} f_{\mathbf{X}}(u,0,0,\ldots,0)u^{d-1}d\mathbf{s}du \\ &= \int_0^r \text{Area}(\mathbb{S}) f_{\mathbf{X}}(u,0,0,\ldots,0)u^{d-1}du. \end{aligned}$$

Diffentiating shows $f_R(r) = \text{Area}(\mathbb{S}) f_{\mathbf{X}}(r,0,0,\ldots,0)r^{d-1}$. Hence for $\mathbf{x} \neq \mathbf{0}$, the radial symmetry shows

$$f_{\mathbf{X}}(\mathbf{x}) = f(|\mathbf{x}|,0,\ldots,0) = \frac{f_R(|\mathbf{x}|)|\mathbf{x}|^{1-d}}{\text{Area}(\mathbb{S})} = \frac{\Gamma(d/2)}{2\pi^{d/2}}|\mathbf{x}|^{1-d} f_R(|\mathbf{x}|).$$

The key fact here is that calculating the density of multivariate $\mathbf{X}$ only requires calculating the univariate function $f_R(r)$.

Therefore, when $\mathbf{X}$ is α-stable with characteristic function (2), the above reasoning shows

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \left(\Gamma(d/2)/\left(2\pi^{d/2}\right)\right)|\mathbf{x}-\boldsymbol{\delta}|^{1-d} f_R(|\mathbf{x}-\boldsymbol{\delta}|\,|\alpha,\gamma_0,d) & \mathbf{x} \neq \boldsymbol{\delta} \\ \Gamma(d/\alpha)/\left(\alpha 2^{d-1}\pi^{d/2}\Gamma(d/2)^2\gamma_0^d\right) & \mathbf{x} = \boldsymbol{\delta}. \end{cases}$$

The value at $\mathbf{x} = \boldsymbol{\delta}$ uses (12). It is useful to consider the *radial function* $h(r|\alpha,d) = f_{\mathbf{X}}(r,0,\ldots,0|\alpha,\gamma_0 = 1,\boldsymbol{\delta} = \mathbf{0})$, which is given by

$$h(r|\alpha,d) = \begin{cases} \Gamma(d/2)/\left(2\pi^{d/2}\right)r^{1-d} f_R(r|\alpha,\gamma_0 = 1,d) & r > 0 \\ \Gamma(d/\alpha)/\left(\alpha 2^{d-1}\pi^{d/2}\Gamma(d/2)^2\right) & r = 0. \end{cases}$$

Then for a general isotropic α-stable $\mathbf{X}$ with scale γ_0 and location $\boldsymbol{\delta}$,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\gamma_0^d} h\left(\frac{|\mathbf{x}-\boldsymbol{\delta}|}{\gamma_0}\,\middle|\,\alpha,d\right). \tag{13}$$

3 Elliptically contoured stable distributions

3.1 Densities of elliptically contoured stable laws

Let $\mathbf{Y}$ be d-dimensional α-stable elliptically contoured random vector with shape matrix Σ and shift vector $\boldsymbol{\delta}$. Then $\mathbf{Y} \overset{d}{=} A^{1/2}\mathbf{G} + \boldsymbol{\delta}$, where positive $A \sim \mathbf{S}(\alpha/2, 1, 2(\cos\pi\alpha/4)^{2/\alpha}, 0)$ and $\mathbf{G} \sim \text{N}(0,\Sigma)$ as above. It is well known that

$\mathbf{G} \stackrel{d}{=} \Sigma^{1/2}\mathbf{G}_1$, where $\Sigma^{1/2}$ is from the Cholesky decomposition of Σ and $\mathbf{G}_1 \sim$N(0,I) has independent standard normal components. Hence $\mathbf{Y} \stackrel{d}{=} A^{1/2}\Sigma^{1/2}\mathbf{G}_1 + \boldsymbol{\delta} = \Sigma^{1/2}A^{1/2}\mathbf{G}_1 + \boldsymbol{\delta} := \Sigma^{1/2}\mathbf{X} + \boldsymbol{\delta}$, where $\mathbf{X}$ is radially symmetric α-stable. So $\mathbf{Y}$ is an affine transformation of $\mathbf{X}$, and (13) shows

$$f_{\mathbf{Y}}(\mathbf{y}) = |\det \Sigma|^{-1/2} f_{\mathbf{X}}(\Sigma^{-1/2}\mathbf{y}) = |\det \Sigma|^{-1/2} h\left(|\Sigma^{-1/2}(\mathbf{y} - \boldsymbol{\delta})| \,\middle|\, \alpha, d \right). \tag{14}$$

3.2 Statistical analysis of data as elliptical stable

We first describe ways of assessing a d-dimensional data set to see if it is approximately sub-Gaussian and then estimating the parameters of a sub-Gaussian vector. These methods are illustrated using the 30 stocks that make up the Dow Jones index.

First perform a one dimensional stable fit to each coordinate of the data using one of the univariate estimation methods to get estimates $\hat{\boldsymbol{\theta}}_i = (\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i, \hat{\delta}_i)$. If the α_i's are significantly different, then the data is not jointly α-stable, so it cannot be sub-Gaussian. Likewise, if the β_i's are not all close to 0, then the distribution is not symmetric and it cannot be sub-Gaussian. If the α_i's are all close, form a pooled estimate of $\alpha = (\sum_{i=1}^{d} \alpha_i)/d =$ average of the indices of each component. Then shift the data by $\hat{\boldsymbol{\delta}} = (\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_d)$ so the distribution is centered at the origin.

Next, assess for elliptical behavior. This can be approached by examining two dimensional projections because of the following result. If $\mathbf{X}$ is a d-dimensional elliptical α-stable random vector, then every two dimensional projection

$$\mathbf{Y} = (Y_1, Y_2) = (\mathbf{a}_1^T\mathbf{X}, \mathbf{a}_2^T\mathbf{X}) \tag{15}$$

($\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^d$) is a 2-dimensional elliptical α-stable random vector. Conversely, suppose $\mathbf{X}$ is a d-dimensional α-stable random vector with the property that every two dimensional projection of form (15) is non-singular elliptical, then d-dimensional $\mathbf{X}$ is non-singular elliptical α-stable. Thus is suffices to assess multivariate data by looking at two dimensional distributions. While one cannot do this for all projections, one can check pairs visually by looking at scatter plots.

Estimating the $d(d+1)/2$ parameters (upper triangular part) of Σ can be done in two ways. Note that for any $\mathbf{u}$, $\mathbf{u}^T\mathbf{X}$ is univariate α-stable with scale $\gamma(\mathbf{u}) = (\mathbf{u}^T\Sigma\mathbf{u})^{1/2}$. For the first method, set $r_{ii} = \gamma_i^2$, i.e. the square of the scale parameter of the i-th coordinate. Then estimate r_{ij} by analyzing the pair (X_i, X_j) and take $r_{ij} = (\gamma^2(1,1) - r_{ii} - r_{jj})/2$, where $\gamma(1,1)$ is the scale parameter of

$(1,1)^T(X_i, X_j) = X_i + X_j$. This involves estimating $d + d(d-1)/2 = d(d+1)/2$ one dimensional scale parameters.

For the second method, note that $E\exp(i\mathbf{u}^T\mathbf{X}) = \exp(-\gamma(\mathbf{u})^\alpha)$, so

$$[-\ln E\exp(i\mathbf{u}^T\mathbf{X})]^{2/\alpha} = \mathbf{u}^T\Sigma\mathbf{u} = \sum_i u_i^2\sigma_{ii} + 2\sum_{i<j} u_i u_j \sigma_{ij}.$$

This is a linear function of the σ_{ij}'s, so they can be estimated by regression. This method may be more accurate because it uses multiple directions, whereas the first method uses only three directions: (1,0), (0,1) and (1,1). Sample estimates of $\gamma^2(\mathbf{u})$ on a grid of $\mathbf{u}$ points can be used for the middle term above. In both methods, checks should be made to test that the resulting matrix Σ is positive definite.

Adjusted daily closing prices for the 30 stocks in the current Dow Jones index were collected between January 3, 2000 and December 31, 2004. Days with missing prices for one or more stock were deleted - this occurred 8 times in the 2256 trading days. Log-ratios of consecutive prices were computed separately for each stock, with the resulting data set having 2247 returns for 30 stocks.

The results of the analysis of each component of the Dow Jones data set is given in Table 1, Figure 2 shows plots of the estimated α and β for each of the 30 components from the table, and Figure 3 shows one pairwise plot. While there seems to be noticeable variability in the α's and some β differ from 0, we argue below that the stable model gives a better fit than a normal model and proceed with the analysis. If we want to use an elliptical multivariate distribution, allow for heavy tails, and retain the property of accumulated returns having the same type of distribution as daily returns, then one has to use an elliptical stable model.

The 30×30 shape matrix Σ was estimated for this set using the first method above. For space reasons we do not show this large matrix, instead a heat map of Σ is displayed in Figure 4. The color shows the size of the entries in the shape matrix. The estimation of the individual stable fits and the shape matrix estimation using maximum likelihood estimation for the 30 component example took 175 seconds on a desktop PC.

It is possible to compute the log-likelihood ratio for the stable elliptical fit vs. a multivariate normal fit. For the Dow Jones data, the stable log-likelihood is $\ell_1 = 96307$. In contrast, if the data is fit with a $\mathrm{N}(\boldsymbol{\mu}, \Sigma)$ model, the log-likelihood is $\ell_2 = 97549$. The likelihood ratio test is $\exp(\ell_1 - \ell_2) \approx 10^{539}$, strongly favoring the stable model.

Because it is possible to quickly simulate from an elliptical stable distribution of high dimension, Monte Carlo estimates of probabilities can be computed. Figure 5 compares the probability $P(|X_i| < a, i = 1, \ldots, 30)$ for (a) the observed data of size 1247, (b) a MC estimate from a simulated stable sample of size $n = 10,000$

symbol	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$
MMM	1.69	0.27	0.00972	-0.000657
AA	1.86	0.16	0.01623	-0.000576
MO	1.53	-0.05	0.01067	0.001335
AXP	1.72	-0.00	0.01383	0.000284
AIG	1.69	0.05	0.01168	-0.000313
BA	1.80	-0.05	0.01379	0.000720
CAT	1.82	0.27	0.01334	-0.000213
C	1.70	0.02	0.01280	0.000210
KO	1.62	0.01	0.00956	-0.000222
DD	1.69	0.26	0.01134	-0.001330
XOM	1.79	-0.25	0.00966	0.000862
GE	1.73	0.11	0.01268	-0.000670
GM	1.71	0.09	0.01319	-0.000709
HPQ	1.68	0.05	0.01805	-0.000970
HD	1.65	0.01	0.01436	-0.000229
HON	1.64	0.08	0.01460	-0.000469
INTC	1.75	0.05	0.02048	-0.000495
IBM	1.59	0.02	0.01179	-0.000361
JNJ	1.73	0.02	0.00939	0.000376
JPM	1.67	-0.00	0.01452	-0.000313
MCD	1.69	-0.03	0.01128	0.000115
MRK	1.72	-0.10	0.01127	0.000218
MSFT	1.65	0.02	0.01386	-0.000498
PFE	1.75	0.00	0.01216	0.000095
PG	1.55	0.08	0.00816	0.000170
SBC	1.71	0.01	0.01339	-0.000474
UTX	1.77	-0.01	0.01263	0.000721
VZ	1.76	0.11	0.01274	-0.000637
WMT	1.66	0.08	0.01185	-0.000689
DIS	1.79	0.16	0.01485	-0.000644
	$\overline{\alpha}$=1.71			

Table 1: Maximum likelihood estimates of stable parameters for the 30 stocks in the Dow Jones index. Note that δ is in the continuous 0-parameterization. α and β across stocks are plotted in Figure 2.

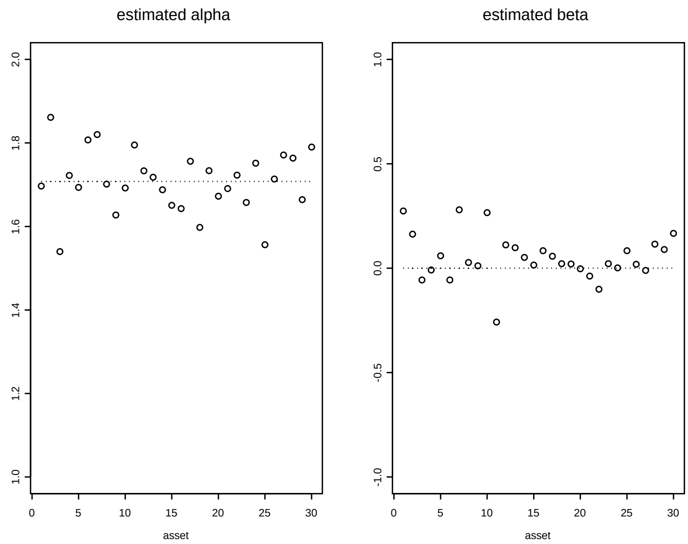

Figure 2: α and β for the 30 components of the Dow Jones data. Order is as shown in Table 1.

generated from the elliptical stable fit, and (c) a MC estimate from a simulated normal sample of size $n = 10,000$ generated from the normal fit. Note that the normal fit severely underestimates the tail, while the stable fit underestimates the tail for quantiles less than 0.98, and overestimate the tail probability for higher quantiles.

Finally, we note that most of what we have done here is easily extended to other elliptical distributions, e.g. multivariate t-distributions with elliptical contours. In all cases, the amplitude function of the isotropic case gives a way to evaluate multivariate densities and bivariate projections can be used to assess for multivariate elliptical shape. The calculations described above are now part of the program STABLE, Robust Analysis Inc (2005).

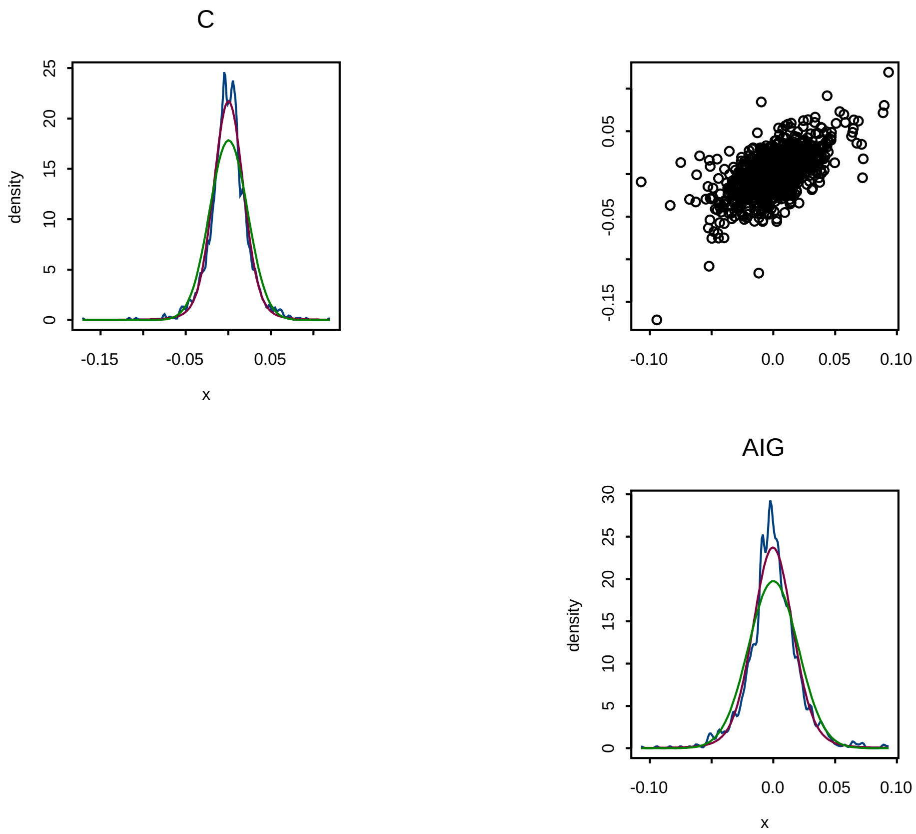

Figure 3: Comparison of returns for AIG and Citigroup (symbol C). The scatterplot shows an approximate elliptical pattern. The other plots show the marginals for each asset: the blue curve is smoothed data, red curve is stable fit, green curve is normal fit.

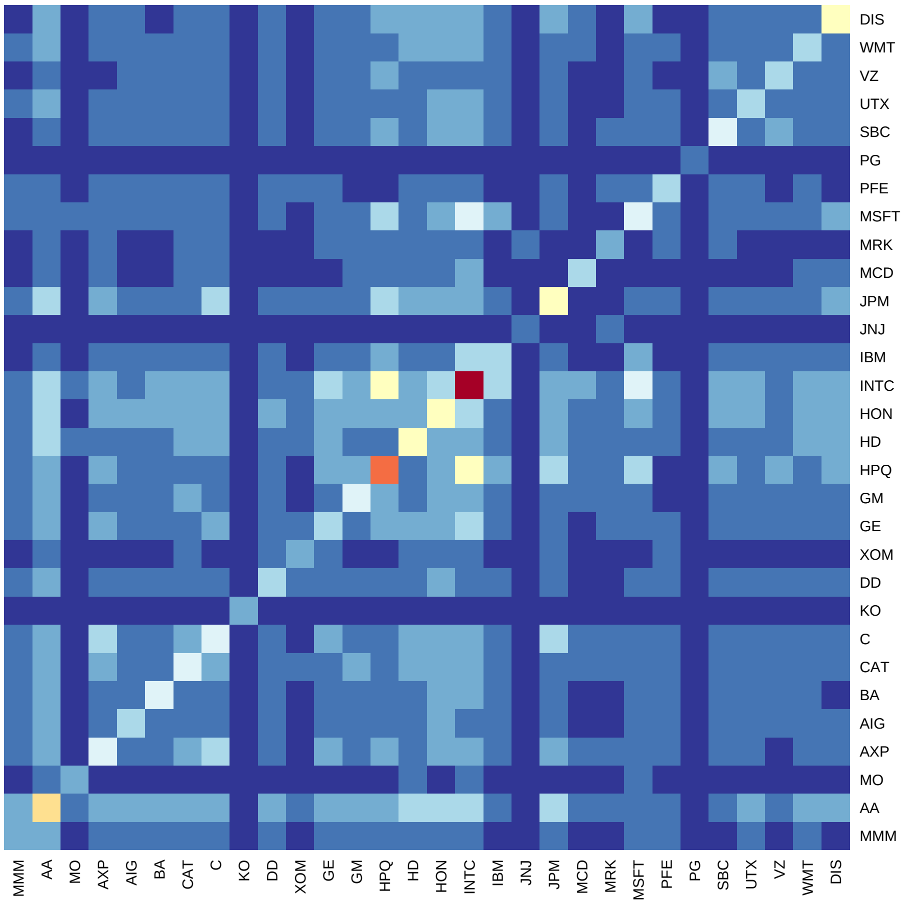

Figure 4: Heat map of the shape matrix Σ. Blue colors corresponds to low values (min=0.000017), to white, to yellow, to orange, to red for high values (max=0.000419).

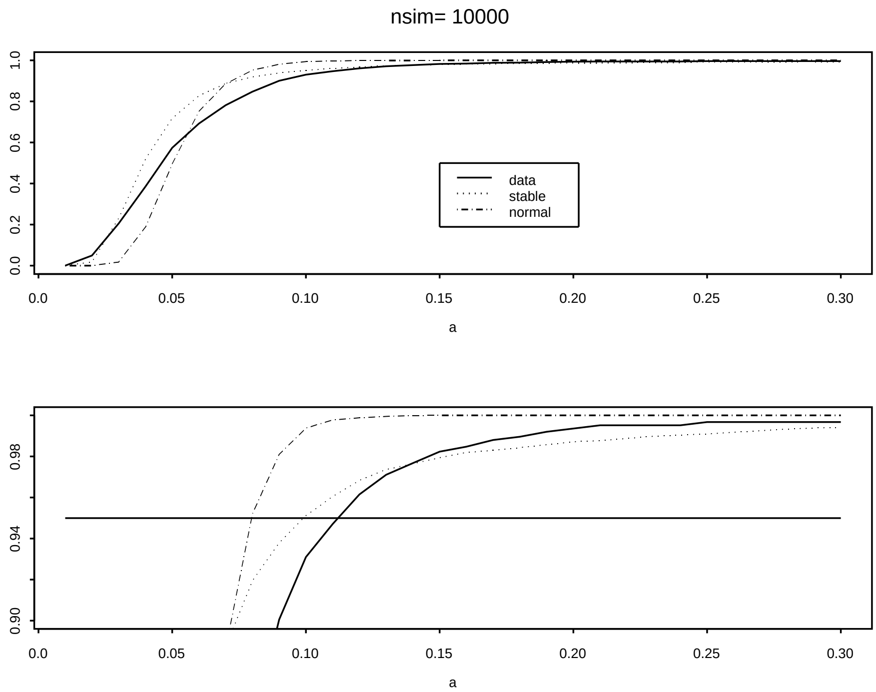

Figure 5: Comparison of empirical estimate of $P(|X_i| < a, i = 1, \ldots, 30)$ (solid line) for 30 dimensional data set to Monte Carlo estimates for stable fit (dotted line) and normal fit (dash-dot line). The top plot uses the full range of [0,1] for the vertical scale, the bottom plot restricts the vertical scale to [0.9,1]. A horizontal line at $p = 0.95$ is added to the bottom graph for reference.

A More facts about the amplitude distribution

There are many facts about the amplitude density and d.f. Since they are useful in finance applications, in signal processing (see Kuruoglu and Zerubia (2004)), and astronomy, we collect them here.

Using the series expansions for stable densities in equations (4) and (5) leads to series expansions for $f_R(r)$ and $F_R(r)$: when $0 < \alpha < 1$

$$F_R(r) = 1 - \frac{2}{\pi\alpha\Gamma(d/2)} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\Gamma\left(\frac{k\alpha+2}{2}\right)\Gamma\left(\frac{k\alpha+d}{2}\right)\sin\left(\frac{k\alpha\pi}{2}\right)}{k\,k!} \left(\frac{r}{2\gamma_0}\right)^{-k\alpha} \quad (16)$$

$$f_R(r) = \frac{1}{\pi\gamma_0\Gamma(d/2)} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\Gamma\left(\frac{k\alpha+2}{2}\right)\Gamma\left(\frac{k\alpha+d}{2}\right)\sin\left(\frac{k\alpha\pi}{2}\right)}{k!} \left(\frac{r}{2\gamma_0}\right)^{-k\alpha-1} \quad (17)$$

When $1 < \alpha < 2$,

$$F_R(r) = \frac{4}{\alpha\Gamma(d/2)} \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma\left(\frac{2k+d}{\alpha}\right)}{(2k+d)\,k!\,\Gamma\left(\frac{2k+d}{2}\right)} \left(\frac{r}{2\gamma_0}\right)^{2k+d} \quad (18)$$

$$f_R(r) = \frac{2}{\alpha\gamma_0\Gamma(d/2)} \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma\left(\frac{2k+d}{\alpha}\right)}{k!\,\Gamma\left(\frac{2k+d}{2}\right)} \left(\frac{r}{2\gamma_0}\right)^{2k+d-1} \quad (19)$$

When $\alpha < 1$, (16) and (17) converges absolutely for any $r > 0$; when $\alpha > 1$, they are asymptotic series as $r \to \infty$. Likewise, (18) and (19) are absolutely convergent for $\alpha > 1$ and an asymptotic series for $\alpha < 1$ for r near 0.

Let $f_d(r) = f_{R,d}(r)$ be the amplitude density and $F_d(r) = F_{R,d}(r)$ be the amplitude d.f. in d dimensions. An argument using (6) and (7) shows

$$F_{d+2}(r) = F_d(r) - \frac{r}{d} f_d(r) \quad \text{and} \quad d f_{d+2}(r) = (d-1) f_d(r) - r f_d'(r). \quad (20)$$

One consequence of the latter expression is that the score function for R can be computed without explicitly differentiating:

$$-\frac{d}{dr}\log f_d(r) = -\frac{f_d'(r)}{f_d(r)} = \frac{d-1}{r} - \frac{d\,f_{d+2}(r)}{r f_d(r)}.$$

When $\alpha = 2$, $R^2 = X_1^2 + \cdots + X_d^2 = 2\gamma_0^2 T$, where T is chi-squared with d degrees of freedom. The d.f. and density are $F_R(r) = F_T(r^2/(2\gamma_0^2)) = 1 - \Gamma(d/2, r^2/(4\gamma_0^2))/\Gamma(d/2)$ and $f_R(r) = (r/\gamma^2)\ f_T(r^2/(2\gamma_0^2))$. In two dimensions, $R = \sqrt{2}\gamma_0\sqrt{T}$ is a Rayleigh distribution with density and d.f.

$$f_R(r) = \frac{1}{2\gamma_0^2} r e^{-r^2/(4\gamma_0^2)} \quad \text{and} \quad F_R(r) = 1 - e^{-r^2/(4\gamma_0^2)}. \quad (21)$$

(Note that this is not the customary scaling for the Rayleigh, which is based on $\mathbf{X} \sim N(0, \gamma_0^2 I)$ and has density $r/\gamma_0^2 \exp(-r^2/(2\gamma_0^2))$ and d.f. $1 - \exp(-r^2/(2\gamma_0^2))$.)

When $\alpha = 1$, the amplitude density and d.f. have explicit formula in all dimensions. The expressions in dimensions 1, 2 and 3 are:

$$\begin{array}{lll} d = 1 & f_R(r) = \frac{2}{\pi}\gamma_0/(\gamma_0^2 + r^2) & F_R(r) = \frac{2}{\pi}\arctan(r/\gamma_0) \\ d = 2 & f_R(r) = \gamma_0 r/(\gamma_0^2 + r^2)^{3/2} & F_R(r) = 1 - \gamma_0/\sqrt{\gamma_0^2 + r^2} \\ d = 3 & f_R(r) = \dfrac{4\gamma_0}{3\pi}\dfrac{\gamma_0^2 + 2r^2}{(\gamma_0^2 + r^2)^2} & F_R(r) = \dfrac{2}{\pi}\left[\arctan(r/\gamma_0) - \dfrac{\gamma_0 r}{3(\gamma_0^2 + r^2)}\right] \end{array}$$

Expressions for higher dimensions can be found using the recursion relations (20).

The fractional moments of R can be found using (3): if $-d < p < \alpha$,

$$\begin{aligned} E(R^p) &= E|\mathbf{X}|^p = E(AT)^{p/2} = (EA^{p/2})(ET^{p/2}) \\ &= (2\gamma_0)^p \frac{\Gamma(1 - p/\alpha)}{\Gamma(1 - p/2)} \frac{\Gamma((d+p)/2)}{\Gamma(d/2)}, \end{aligned} \tag{22}$$

where the first expectation (which is finite for all for all $p < \alpha$) is from Section 2.1 of Zolotarev (1986); a short calculation is used for the second expectation (which is finite for all $p > -d$). This expression holds for complex p in the strip $-d < \mathbf{Re}\, p < \alpha$, giving the Mellin transform of R.

The above expression for moments combined with Markov's inequality gives a uniform bound on tail probabilities of R and isotropic $\mathbf{X}$:

$$\sup_{r>0} r^p(1 - F_R(r)) = \sup_{r>0} r^p P(|\mathbf{X}| > r) \le E(R^p), \quad 0 < p < \alpha \tag{23}$$

Let X be univariate strictly stable, e.g. $X \sim \mathbf{S}(\alpha, \beta, \gamma, 0)$ with $\alpha \neq 1$ or $X \sim \mathbf{S}(1, 0, \gamma, 0)$. Section 3.6 of Zolotarev (1986) shows $\log|X|$ has mean and variance

$$\begin{aligned} E(\log|X|) &= \gamma_{Euler}\left(\frac{1}{\alpha} - 1\right) + \log\left(\frac{\gamma}{(\cos\alpha\theta_0)^{1/\alpha}}\right) \\ \mathrm{Var}(\log|X|) &= \frac{\pi^2(1 + 2/\alpha^2)}{12} - \theta_0^2 \end{aligned}$$

where $\gamma_{Euler} \approx 0.57721$ is Euler's constant and $\theta_0 = \alpha^{-1}\arctan(\beta\tan(\pi\alpha/2))$. (Note the constant θ_0 arises above because Zolotarev uses a different parameterization.) The following is a multivariate generalization of this result, it uses the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Lemma 1 $\log R$ *has moment generating function* $E \exp(u \log R) = E\, R^u$ *given by (22) for* $-d < u < \alpha$*. The mean and variance of* $\log R$ *are*

$$\begin{aligned} E(\log R) &= \log(2\gamma_0) + \gamma_{Euler}\left(\frac{1}{\alpha} - \frac{1}{2}\right) + \frac{1}{2}\psi(d/2) \\ \mathrm{Var}(\log R) &= \frac{\pi^2}{6}\left(\frac{1}{\alpha^2} - \frac{1}{4}\right) + \frac{1}{4}\psi'(d/2). \end{aligned}$$

We will not pursue it here, but there are several ways of estimating γ_0 and α from amplitude data: (a) maximum likelihood estimation using $f_R(r)$, (b) fractional moment methods using (22), and (c) using the first and second moments of $\log R$ and Lemma 1.

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