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E – companion to:

Decision Analysis with Geographically Varying Outcomes: Preference Models and Illustrative Applications

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APPENDIX A: PREFERENTIAL INDEPENDENCE.

In decision analysis, specifying a multiattribute value or utility function requires assessments from a decision maker, and this can be difficult because it requires the determination of an *n*-dimensional function. To simplify this, researchers have established conditions on preferences under which the form of the value or utility function is simplified. One of these conditions is *preferential independence*. A subset of $Z_1, Z_2, \ldots Z_n$, when $n \geq 3$, is defined to be preferentially independent of its complement if the rank ordering of alternatives with no uncertainty that have common levels for the complementary attributes does not depend on those common levels. If this property holds for all subsets of $Z_1, Z_2, \ldots, Z_n$, then *mutual preferential independence* is said to hold, and in this case

$$V(z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} a_i v_i(z_i), \qquad \text{(A-1)}$$

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where $v_i(z_i)$ is called the single attribute value function over z_i, and $a_i \geq 0$ is called the weighting constant for the i^{th} attribute (Debreu 1960). Gorman (1968) derives several conditions related to mutual preferential independence. In particular, it follows from his results that if every *pair* of attributes is preferentially independent of the remaining attributes then mutual preferential independence holds, or, even more specifically, if $\{Z_i, Z_{i+1}\}$ is preferentially independent of its complement for $i = 1, 2, \ldots, n-1$, then mutual preferential independence holds. These results can substantially reduce the number of assessments that must be made in applications to establish that (A-1) is valid. See Keeney and Raiffa (1976 Sections 3.6.2- 3.6.4) for more details.

APPENDIX B: ADDITIONAL DISCUSSION OF THEOREM 1.

Let $\succsim$ ("at least as preferred as") be a preference relation over $\boldsymbol{Z}$, which is the set of possible consequences (where a consequence is a vector of levels across all subregions), with the corresponding strict relation $\succ$ ("more preferred than") and indifference relation $\sim$ ("equally preferred to"). Let $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$ denote arbitrary vectors in $\boldsymbol{Z}$. For $\boldsymbol{z} \in \boldsymbol{Z}$, we define $\boldsymbol{z}_{\bar{i}}$ as the vector of attribute levels in all subregions except subregion i, and $\boldsymbol{z}_{\bar{i}\bar{j}}$ as the vector of attribute levels in all subregions except subregions i and j.

Consider the following conditions on $\succsim$:

(a) Completeness: It must be true that $\boldsymbol{q} \succsim \boldsymbol{r}$ or $\boldsymbol{r} \succsim \boldsymbol{q}$. (That is, any consequences can be compared.)

(b) Transitivity: If $\boldsymbol{q} \succsim \boldsymbol{r}$ and $\boldsymbol{r} \succsim \boldsymbol{s}$, then $\boldsymbol{q} \succsim \boldsymbol{s}$.

(c) Continuity: If $\boldsymbol{q} \succ \boldsymbol{r}$, then there exists $\Delta > 0$ such that for any $\boldsymbol{z} \in \boldsymbol{Z}$, $\max_i |z_i - q_i| < \Delta$ implies $\boldsymbol{z} \succ \boldsymbol{r}$, and $\max_i |z_i - r_i| < \Delta$ implies $\boldsymbol{q} \succ \boldsymbol{z}$.

(d) Dependence on each subregion: For each subregion i, there exist $(q_i, \boldsymbol{z}_{\bar{i}})$ and $(r_i, \boldsymbol{z}_{\bar{i}})$ such that

$\left(q_i, \boldsymbol{z}_{\bar{i}}\right) \succ \left(r_i, \boldsymbol{z}_{\bar{i}}\right)$.

(e) Pairwise spatial preferential independence: For any two subregions *i* and *j*, $\left(q_i, q_j, \boldsymbol{s}_{\overline{ij}}\right) \succsim \left(r_i, r_j, \boldsymbol{s}_{\overline{ij}}\right)$ for some $\boldsymbol{s}_{\overline{ij}}$ implies $\left(q_i, q_j, \boldsymbol{z}_{\overline{ij}}\right) \succsim \left(r_i, r_j, \boldsymbol{z}_{\overline{ij}}\right)$ for all $\boldsymbol{z}_{\overline{ij}}$, $\boldsymbol{z} \in \boldsymbol{Z}$.

This condition implies mutual preferential independence, as described above. Since the independence described here is for a single attribute across multiple subregions, it is labeled "spatial" to distinguish it from preferential independence across multiple attributes, which is considered in Theorem 2.

DEFINITION A-1. An attribute level z^{mid} is called a *tradeoffs midvalue* of z' and z'' in subregion *i* if there exist vectors $\boldsymbol{q}_{\bar{i}}$ and $\boldsymbol{r}_{\bar{i}}$ such that $\left(z', \boldsymbol{q}_{\bar{i}}\right) \sim \left(z^{mid}, \boldsymbol{r}_{\bar{i}}\right)$ and $\left(z^{mid}, \boldsymbol{q}_{\bar{i}}\right) \sim \left(z'', \boldsymbol{r}_{\bar{i}}\right)$.

(f) Homogeneity: If z^{mid} is a tradeoffs midvalue of z' and z'' in subregion *i*, then for any subregion *j* and vectors $\boldsymbol{q}_{\bar{j}}$, $\boldsymbol{r}_{\bar{j}}$ such that $\left(z', \boldsymbol{q}_{\bar{j}}\right) \sim \left(z^{mid}, \boldsymbol{r}_{\bar{j}}\right)$ it must be true that $\left(z^{mid}, \boldsymbol{q}_{\bar{j}}\right) \sim \left(z'', \boldsymbol{r}_{\bar{j}}\right)$. (An equivalent definition is that when two amounts *z*, *z'* have a tradeoffs midvalue in a subregion *i* and also in a subregion *j*, then they have the same tradeoffs midvalue in subregion *i* as in subregion *j*.)

A value function exists over the set of consequences if and only if $\succsim$ is complete, transitive, and continuous, as specified in conditions (a)-(c). (This is a special case of Debreu (1954, 1964), who uses a more general topological space.) It follows directly from Debreu's (1960) theorem that for a region with three or more subregions, a value function can be written as:

$$V(z_1, z_2, \ldots, z_m) = \sum_{i=1}^{m} a_i v_i(z_i) \qquad \text{(A-2)}$$

if and only if conditions (a)-(e) above hold, where v_i is a single attribute value function over z_i, and $a_i > 0$ is the weight associated with subregion *i*. While (A-2) has the same form as (A-1), the z_i in (A-2) represent the levels of the same attribute in different subregions, rather than the levels of different attributes $Z_1, Z_2, \ldots, Z_n$ as in (A-1). Harvey (1986, p. 1126, Condition E) defines the Equal Tradeoffs

Comparisons condition, and he notes that a condition that is mathematically equivalent to homogeneity (our condition (f)) is equivalent to his Equal Tradeoffs Comparisons condition. He then shows (Harvey, 1986, p. 1136-37, proof of Theorem 1) that an intertemporal analog of (1) is a valid representation of preferences if and only if (A-2) and his condition E hold. Since his Condition E is equivalent to our condition (f), that proof also establishes our Theorem 1.

Value functions are only determined to within a positive monotonic transformation, so any positive monotonic transformation of the value function forms in all of the Theorems and Conjectures will also satisfy the specified conditions. See Keeney and Raiffa (1976, Section 3.3.4) and Kirkwood (1997, Section 9.2) for further discussion of this point.

With a stronger condition in place of condition (f), a special case of equation (1) holds where $v(z_i)$ is linear. Consider the stronger condition of *tradeoffs neutrality*: for any subregion *i*, attribute levels z_i' and z_i'', and vectors $\boldsymbol{q}_{\bar{i}}$ and $\boldsymbol{r}_{\bar{i}}$, $\left(z_i', \boldsymbol{q}_{\bar{i}}\right) \sim \left(z^*, \boldsymbol{r}_{\bar{i}}\right)$, implies $\left(z^*, \boldsymbol{q}_{\bar{i}}\right) \sim \left(z_i'', \boldsymbol{r}_{\bar{i}}\right)$, where $z^* = 0.5\left(z_i' + z_i''\right)$. This condition ensures that single-attribute value functions will be linear by the following argument: Since tradeoffs neutrality is a special case of condition (f), equation (1) still holds in this case. By direct substitution, when tradeoffs neutrality holds, $v[0.5 \times (z_i' + z_i'')] = 0.5 \times [v(z_i') + v(z_i'')]$ for any z_i' and z_i''. This is Jensen's equation (Small, 2007, Section 2.3), and the solution is $v(z) = az + b$ for constants a and b when $v(z_i)$ is continuous.

(An anonymous reviewer provided valuable suggestions that helped us to develop the material just presented in this section. This same reviewer also provided many valuable suggestions for the other theorem and conjecture presentations in this Appendix.)

APPENDIX C: ADDITIONAL DISCUSSION OF CONJECTURE 1.

Let $\succsim$ have corresponding strict and indifference relations as previously defined, and let ***q***, ***r***, and ***s*** denote arbitrary consequences. Consider the following conditions on $\succsim$:

(a') Completeness: It must be true that $\boldsymbol{q} \succsim \boldsymbol{r}$ or $\boldsymbol{r} \succsim \boldsymbol{q}$. (That is, any consequences can be compared.)

(b') Transitivity: If $\boldsymbol{q} \succsim \boldsymbol{r}$ and $\boldsymbol{r} \succsim \boldsymbol{s}$, then $\boldsymbol{q} \succsim \boldsymbol{s}$.

(c') Continuity: If $\boldsymbol{q} \succ \boldsymbol{r}$, then there exists $\Delta > 0$ such that for any consequence $\boldsymbol{z}$, $\max_{x,y} \left| z(x,y) - q(x,y) \right| < \Delta$ implies $\boldsymbol{z} \succ \boldsymbol{r}$, and $\max_{x,y} \left| z(x,y) - r(x,y) \right| < \Delta$ implies $\boldsymbol{q} \succ \boldsymbol{z}$.

(d') Dependence on any subregion: For any proper subregion P of the region with area greater than zero, there exist consequences $\boldsymbol{q}$ and $\boldsymbol{r}$ such that $q(x,y) = r(x,y)$ for all $(x,y) \notin P$, and $\boldsymbol{q} \succ \boldsymbol{r}$.

(e') Spatial preferential independence: For any proper subregion P of the region with area greater than zero, if there exist consequences $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{q}'$, and $\boldsymbol{r}'$ such that $\boldsymbol{q} \succsim \boldsymbol{r}$, $q'(x,y) = q(x,y)$ and $r'(x,y) = r(x,y)$ for all $(x,y) \in P$, and $q(x,y) = r(x,y)$ and $q'(x,y) = r'(x,y)$ for all $(x,y) \notin P$, then it must be true that $\boldsymbol{q}' \succsim \boldsymbol{r}'$.

DEFINITION A-2. Consider a proper subregion P of the region with area greater than zero, and consider two consequences $z'(x,y)$ and $z''(x,y)$ that have constant levels z' and z'', respectively, within *P*. An attribute level z^{mid} is a *tradeoffs midvalue* for z' *and* z'' *with respect to P* if there exist functions $q(x,y)$ and $r(x,y)$ defined outside *P* such that the following two indifference relations hold: i) a consequence that is equal to z' within *P* and equal to $q(x,y)$ outside *P* is indifferent to a consequence that is equal to z^{mid} within *P* and equal to $r(x,y)$ outside *P*, and ii) a consequence that is equal to z^{mid} within *P* and equal to $q(x,y)$ outside *P* is indifferent to a consequence that is equal to z'' within *P* and equal to $r(x,y)$ outside *P*.

(f') Homogeneity: If z^{mid} is a tradeoffs midvalue for z' and z'' with respect to a specified proper subregion P of the region with area greater than zero, then it is also a tradeoffs midvalue for z' and z'' with respect to any other proper subregion for which a tradeoffs midvalue for z' and z'' exists.

Note that tradeoffs midvalues are unlikely to exist with respect to very large subregions, since the required $q(x, y)$ and $r(x, y)$ may not exist for such cases. However, for the purposes of both practical applications and possible proofs of Conjecture 1, we are concerned only with subregions which are small relative to the entire region.

We present a plausibility argument for why conditions (a') through (f') might imply (3), and discuss where difficulties arise in the proof. The conditions stated in Conjecture 1 are analogous to those used in Theorem 1, and (3) is analogous to (1). Conditions (a')-(e') are straightforward adaptations of conditions (a)-(e). (Condition (e') is analogous to mutual preferential independence, rather than pairwise preferential independence, because in the non-discrete case there are not existing discrete subregions with which to create pairs.) Condition (f') is analogous to condition (f).

To establish the plausibility of Conjecture 1, start with (1) and define $\lambda_i \equiv a_i / A_i$ in (1), where A_i is the area (for example, in square miles) of subregion *i*. Then (1) can be rewritten as

$$V(z_1, z_2, \ldots, z_m) = \sum_{i=1}^{m} \lambda_i A_i v(z_i) \,. \tag{A-3}$$

Since conditions (a') through (f') are analogous to conditions (a) through (f) required for (1), it is plausible to use (1) as a starting point for developing (3). Extend (A-3) to an attribute that varies over the region as follows: Partition the region into a uniform grid, where the two dimensions of the grid are designed by *x* and *y*, and where the *x* and *y* dimensions of each cell in the grid are designated by Δx and Δy, respectively, so that the area A_i of any cell is $\Delta x \times \Delta y$. While it is easiest to visualize this partition if *A* is rectangular, the analysis can be extended to any region that is bounded by a piecewise smooth curve, as established in the references given below. If λ_i and $v_i(z_i)$ did not vary within a grid cell, then if the assumptions for Theorem 1 are assumed to hold, (A-3) can be written as

$$V(z_1, z_2, \ldots, z_m) = \sum_{i=1}^{m} \lambda(x_i, y_i) v[z(x_i, y_i)] \Delta x \Delta y, \tag{A-4}$$

where x_i and y_i designate some specified but arbitrary point within grid cell *i*, $\lambda(x_i, y_i) \equiv \lambda_i$, and $v[z(x_i, y_i)] \equiv v(z_i)$. Equation (A-4) has the form of a special case of a Riemann sum of $\lambda(x, y)v[z(x, y)]$ over *A*, and if λ and v are both bounded, then their product is also bounded. If these functions are continuous almost everywhere (that is, except on a subset of *A* with measure zero), then $\lambda(x, y)v[z(x, y)]$ is Riemann integrable over *A* if the boundary of *A* is a piecewise smooth curve. (For proofs of this, see Apostol 1962, Section 2.12, or Trench 2003, Theorem 7.1.19.) If $\lambda(x, y)v[z(x, y)]$ is integrable, the Riemann sum in (A-4) will converge to a unique value (which by definition is the integral) as the partition of *A* is made finer so that m approaches infinity and both Δx and Δy approach zero. Thus, in the limit, (A-4) would become

$$V(z) = \iint_A \lambda(x, y) v[z(x, y)] dxdy, \tag{A-5}$$

where *V*(*z*) is the value from a decision-making perspective associated with the distribution of the attribute over the region of interest. The converse of the model result would follow by direct substitution from (3). Note that in (3), $\lambda(x, y)$ in (A-5) has been replaced with *a*(*x*, *y*) to make the notation more parallel to (1). However, the units for *a* in (1) and (3) are different.

The plausibility argument given above is not a proof, and there are two primary difficulties involved in proving Conjecture 1. The first is that the partition of the region used in (1) is fixed, as is the number of subregions. Therefore, we cannot be sure that assessed preferences for different partitions, such as the changing partitions in (A-4) as *m* increases, will yield the same single-subregion value function *v*, and hence yield a single common limit for any possible Riemann sum. This means it is not defensible to examine the limit of (1) as *m* goes to infinity, because a common limit of $v(z_i)$ independent of the partition is not guaranteed to exist. In the conditions for Conjecture 1, we address this with the continuity condition (c'). While this is meant to guarantee convergence as the partition becomes finer, we cannot establish that it does this in the manner required for a unique limit to exist.

A second difficulty is establishing that the weighting function $a(x, y)$ is Riemann integrable. It must be true that $v[z(x, y)]$ is Riemann integrable, because $z(x, y)$ is Riemann integrable by assumption, and conditions (c') and (d') ensure that *v* is continuous and bounded. The weighting function must be bounded, otherwise a violation of condition (d') would occur. However, it must also be established that the weighting function is continuous almost everywhere, and it is not clear precisely what conditions on the preference relation will guarantee this. Condition (c') establishes the continuity of *v*, but does not impose this property on $a(x, y)$. In practical applications, it is difficult to think of a realistic weighting function that is not continuous almost everywhere, but we are unable to prove that conditions (a') through (f') establish this property.

Harvey and Østerdal (2011) provide additional discussion of the steps necessary to establish a result analogous to (3) in the context of continuous time decisions.

APPENDIX D: ADDITIONAL DISCUSSION OF THEOREM 2.

We thank a reviewer for pointing out that the result in Theorem 2 can be developed in either of two ways: By using results from Gorman (1968) and then applying homogeneity conditions, or by applying preferential independence conditions across attributes to Harvey's (1995) value model. We use the former approach here.

Modify the notation presented earlier so that $\boldsymbol{Z}_i$ now designates the vector of *n* attributes $Z_{i1}, Z_{i2}, \ldots, Z_{in}$ in subregion *i* (referred to as the "attribute vector for subregion *i*"), and $\boldsymbol{Z}^j$ designates the vector $Z_{1j}, Z_{2j}, \ldots, Z_{mj}$ of the j^{th} attribute across the *m* subregions (referred to as the "subregion vector for attribute *j*"), where $\boldsymbol{z}_i$ and $\boldsymbol{z}^j$ are vectors of specific levels of $\boldsymbol{Z}_i$ and $\boldsymbol{Z}^j$, respectively. Let $\succsim$ have corresponding strict preference and indifference relations as previously defined, and let ***q***, ***r***, and ***s*** denote arbitrary consequences. Consider the following conditions on $\succsim$, which are analogous to those used in Theorem 1.

(a'') Completeness: It must be true that $\boldsymbol{q} \succsim \boldsymbol{r}$ or $\boldsymbol{r} \succsim \boldsymbol{q}$. (That is, any consequences can be compared.)

(b'') Transitivity: If $\boldsymbol{q} \succsim \boldsymbol{r}$ and $\boldsymbol{r} \succsim \boldsymbol{s}$, then $\boldsymbol{q} \succsim \boldsymbol{s}$.

(c'') Continuity: If $\boldsymbol{q} \succ \boldsymbol{r}$, then there exists $\Delta > 0$ such that for any $\boldsymbol{z} \in \boldsymbol{Z}$, $\max_{i,j} \left| z_{ij} - q_{ij} \right| < \Delta$ implies $\boldsymbol{z} \succ \boldsymbol{r}$, and $\max_{i,j} \left| z_{ij} - r_{ij} \right| < \Delta$ implies $\boldsymbol{q} \succ \boldsymbol{z}$.

(d'') Dependence on each attribute-subregion combination: For each subregion *i* and attribute *j*, there exist consequences $\left(q_{ij}, \boldsymbol{z}_{\overline{ij}}\right)$ and $\left(r_{ij}, \boldsymbol{z}_{\overline{ij}}\right)$ such that $\left(q_{ij}, \boldsymbol{z}_{\overline{ij}}\right) \succ \left(r_{ij}, \boldsymbol{z}_{\overline{ij}}\right)$.

(e'') Preferential independence:

1) Subregion preferential independence: For any subregion *i*, $\left(\boldsymbol{q}_i, \boldsymbol{s}_{\bar{i}}\right) \succsim \left(\boldsymbol{r}_i, \boldsymbol{s}_{\bar{i}}\right)$ for some $\boldsymbol{s}_{\bar{i}}$ implies $\left(\boldsymbol{q}_i, \boldsymbol{z}_{\bar{i}}\right) \succsim \left(\boldsymbol{r}_i, \boldsymbol{z}_{\bar{i}}\right)$ for all $\boldsymbol{z}_{\bar{i}}$, and

2) Attribute preferential independence: For any attribute *j*, $\left(\boldsymbol{q}^j, \boldsymbol{s}^{\bar{j}}\right) \succsim \left(\boldsymbol{r}^j, \boldsymbol{s}^{\bar{j}}\right)$ for some $\boldsymbol{s}^{\bar{j}}$ implies $\left(\boldsymbol{q}^j, \boldsymbol{z}^{\bar{j}}\right) \succsim \left(\boldsymbol{r}^j, \boldsymbol{z}^{\bar{j}}\right)$ for all $\boldsymbol{z}^{\bar{j}}$.

DEFINITION A-3. Preferences over the region of interest with respect to a set of attribute vectors $\boldsymbol{Z}_1, \ \ldots, \ \boldsymbol{Z}_m$ for subregions 1, …, *m* are *multiattribute homogeneous* if, when two alternatives that differ only in the attribute levels for a specified subregion are indifferent, then the same indifference relation holds for those same attribute levels in any subregion. (In this definition, the scalar attribute for each subregion considered in Theorem 1 is replaced with an attribute vector for each subregion).

(f'') Homogeneity: $\succsim$ is multiattribute homogeneous with respect to $\boldsymbol{Z}_1, \ \ldots, \ \boldsymbol{Z}_m$.

Condition (f'') plays a role for multiple attributes similar to the single-attribute homogeneity condition (f) previously defined for Theorem 1. Condition (f) assumes there is a single attribute that has a tradeoffs midvalue, but when there are multiple attributes there is not an unambiguous meaning for the tradeoffs midvalue. Therefore, we express homogeneity here considering levels of multiple attributes. Conversely, the definition given here requires considerations of tradeoffs between attributes, and therefore is not applicable to the single-attribute consequences used in Theorem 1.

We first show that condition (e”) implies an additive form for $V(z)$. The first step is to show that the set consisting of any pair of attribute-subregion combinations is preferentially independent of its complement. Let Z_{ac} and Z_{bd} represent two such combinations, where *a* and *b* are arbitrary distinct subregions, and *c* and *d* are arbitrary distinct attributes. From condition (e”), each of $\boldsymbol{Z}_a$, $\boldsymbol{Z}_b$, $\boldsymbol{Z}^c$, and $\boldsymbol{Z}^d$ is preferentially independent of its complement. From Theorem 1 in Gorman (1968), the union of $\boldsymbol{Z}_a$ and $\boldsymbol{Z}^d$ is preferentially independent of its complement, as is the union of $\boldsymbol{Z}_b$ and $\boldsymbol{Z}^c$. The intersection of these two unions is $\{Z_{ac}, Z_{bd}\}$, and since both unions are preferentially independent of their complements, Theorem 1 in Gorman also implies that $\{Z_{ac}, Z_{bd}\}$ must be preferentially independent of its complement. Since the choices of *a*, *b*, *c*, and *d* were arbitrary, any pair of attribute-subregion combinations is preferentially independent of its complement for distinct attributes and subregions. It is then straightforward to also use Theorem 1 in Gorman to show that a pair of attribute-subregion combinations is preferentially independent of its complement in the case where either the attribute or the subregion is common to both combinations. Hence, from inductive application of Gorman's results, also stated as a Corollary on page 114 of Keeney and Raiffa (1976), the Z_{ij} are mutually preferentially independent, and therefore:

$$V(z) = \sum_{i=1}^{m}\sum_{j=1}^{n} k_{ij} v_{ij}(z_{ij})\ . \tag{A-6}$$

Since the attributes are multiattribute homogeneous by condition (f’’), it follows from an analogous argument to the one given in the proof of Theorem 1 that single-attribute value functions v_{ij} cannot depend on the subregion *i*, and hence the following equation holds in this case:

$$V(z)=\sum_{i=1}^{m}\sum_{j=1}^{n}k_{ij}v_j(z_{ij})\,. \qquad \text{(A-7)}$$

To show that $k_{ij}=a_i b_j$, and hence (4) holds, first assume without loss of generality that the subregions and attributes are labeled so that the largest scaling constant is k_{11}. Consider two hypothetical alternatives: 1) all the attribute-subregion combinations except Z_{11} and Z_{1j} are set to arbitrary levels, Z_{11} is set to its least preferred level so that $v_1(z_{11})=0$ in (A-7), and Z_{1j} is set to its most preferred level so that $v_j(z_{1j})=1$, and 2) another hypothetical alternative with all the attribute-subregion combinations except Z_{11} and Z_{1j} set to the same arbitrary levels as the first alternative, Z_{1j} set to its least preferred level so $v_j(z_{1j})=0$, and Z_{11} set to the level z_{11}^{j} such that the two alternatives are equally preferred. Then equating the values for each of these two alternatives calculated using (A-7) and cancelling common terms results in $k_{11}v_1(z_{11}^{j})=k_{1j}$ for any $j\neq 1$. However, by condition (f''), if this equation holds for subregion 1, then the same level z_{11}^{j} must make the analogous equation true for any subregion *i*, and hence $k_{i1}v_1(z_{11}^{j})=k_{ij}$ for any *i*. Define $b_j\equiv v_1(z_{11}^{j})$ and $a_i\equiv k_{i1}$. Substituting these definitions into $k_{i1}v_1(z_{11}^{j})=k_{ij}$ gives $k_{ij}=a_i b_j$. Substitute this into (A-7), and (4) follows. The converse of the model result follows by direct substitution from (4).

APPENDIX E: ADDITIONAL DISCUSSION OF CONJECTURE 2.

Let $\succsim$ have corresponding strict and indifference relations as previously defined, and let ***q***, ***r***, and ***s*** denote arbitrary consequences. Let z^j designate a Riemann integrable function such that $z^j(x,y)\in I^j$ for all (x,y) in the region. Let $\boldsymbol{z}^{\bar{j}}$ and $z^{\bar{i}\bar{j}}$ be defined as previously. As in Conjecture 1, assume that the boundary of the region is a piecewise smooth curve. Consider the following conditions on $\succsim$, which are analogous to those used in Theorem 2:

(a''') Completeness: It must be true that $\boldsymbol{q} \succsim \boldsymbol{r}$ or $\boldsymbol{r} \succsim \boldsymbol{q}$. (That is, any consequences can be compared.)

(b''') Transitivity: If $\boldsymbol{q} \succsim \boldsymbol{r}$ and $\boldsymbol{r} \succsim \boldsymbol{s}$, then $\boldsymbol{q} \succsim \boldsymbol{s}$.

(c''') Continuity: If $\boldsymbol{q} \succ \boldsymbol{r}$, then there exists $\Delta > 0$ such that for any consequence $\boldsymbol{z}$, $\max_{x,y,j} \left| z^j(x,y) - q^j(x,y) \right| < \Delta$ implies $\boldsymbol{z} \succ \boldsymbol{r}$, and $\max_{x,y,j} \left| z^j(x,y) - r^j(x,y) \right| < \Delta$ implies $\boldsymbol{q} \succ \boldsymbol{z}$.

(d''') Dependence on any attribute in any subregion: For any attribute j and proper subregion P of the region with area greater than zero, there exist consequences $\boldsymbol{q}$ and $\boldsymbol{r}$ such that $\boldsymbol{q}^{\bar{j}}(x,y) = \boldsymbol{r}^{\bar{j}}(x,y)$ for all (x,y), $\boldsymbol{q}(x,y) = \boldsymbol{r}(x,y)$ for all $(x,y) \notin P$, and $\boldsymbol{q} \succ \boldsymbol{r}$.

(e''') Preferential independence:

1) Spatial preferential independence: For any proper subregion P of the region with area greater than zero, if there exist consequences $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{q}'$, and $\boldsymbol{r}'$ such that $\boldsymbol{q} \succsim \boldsymbol{r}$, $\boldsymbol{q}'(x,y) = \boldsymbol{q}(x,y)$ and $\boldsymbol{r}'(x,y) = \boldsymbol{r}(x,y)$ for all $(x,y) \in P$, and $\boldsymbol{q}(x,y) = \boldsymbol{r}(x,y)$ and $\boldsymbol{q}'(x,y) = \boldsymbol{r}'(x,y)$ for all $(x,y) \notin P$, then it must be true that $\boldsymbol{q}' \succsim \boldsymbol{r}'$.

2) Pairwise attribute preferential independence: For any two attributes i and j, $\left(q^i, q^j, \boldsymbol{s}^{\overline{ij}}\right) \succsim \left(r^i, r^j, \boldsymbol{s}^{\overline{ij}}\right)$ for some $\boldsymbol{s}^{\overline{ij}}$ implies $\left(q^i, q^j, \boldsymbol{z}^{\overline{ij}}\right) \succsim \left(r^i, r^j, \boldsymbol{z}^{\overline{ij}}\right)$ for all $\boldsymbol{z}^{\overline{ij}}$.

DEFINITION A-4. Consider a proper subregion P of the region with area greater than zero, and consider two consequences $\boldsymbol{q}$ and $\boldsymbol{r}$. Preferences over the region are *continuously multiattribute homogeneous* if $\boldsymbol{q} \succsim \boldsymbol{r}$, $\boldsymbol{q}(x,y) = \boldsymbol{r}(x,y)$ for $(x,y) \notin P$, and $\boldsymbol{q}(x,y) = (q_1, \ldots, q_n)$ and $\boldsymbol{r}(x,y) = (r_1, \ldots, r_n)$ for $(x,y) \in P$ where $q_1, \ldots, q_n$ and $r_1, \ldots, r_n$ are constants implies that for any consequences $\boldsymbol{z}'$ and $\boldsymbol{z}''$ and subregion P' such that $\boldsymbol{z}'(x,y) = \boldsymbol{z}''(x,y)$ for $(x,y) \notin P'$, and $\boldsymbol{z}'(x,y) = (q_1, \ldots, q_n)$ and $\boldsymbol{z}''(x,y) = (r_1, \ldots, r_n)$ for $(x,y) \in P'$ then $\boldsymbol{z}' \succsim \boldsymbol{z}''$.

(f''') : Homogeneity: $\succsim$ is continuously multiattribute homogeneous over the region.

The development of Conjecture 2 from Theorem 2 is analogous to the development of Conjecture 1 from Theorem 1. A similar plausibility argument can be given for Conjecture 2, in which a Riemann sum analogous to equation (A-4) is developed. A further development analogous to the Conjecture 1 reasoning leads to (5), and direct substitution from (5) yields the converse of the result. However, analogous difficulties to those involved in proving Conjecture 1 arise here as well.

APPENDIX F: EXTENSIONS OF THE PREFERENCE MODELS TO ADDRESS UNCERTAINTY.

In this Appendix, we examine the case in which the consequences of alternatives are uncertain. We assume that probabilities can be assigned to the possible consequences of each alternative, and we wish to rank alternatives by their overall desirability using their expected utility, computed as the expected value of a *single attribute* (or *multiattribute) utility function*. The preference conditions in Section 3 can be extended to decisions under uncertainty to determine the requirements for an additive utility function. The primary difference is that the preferential independence condition discussed in Section 3 must be replaced by a considerably stronger condition called *additive independence*.

DEFINITION A-5. *Additive independence* with respect to Z over the region of interest holds if the rank ordering for any set of alternatives depends only on the marginal probability distributions for each alternative over the levels $z_1, z_2, \ldots, z_m$ of Z in each of the subregions 1, ..., m.

We first consider the case analogous to Theorem 1, in which the level for the single attribute Z does not vary within each subregion, and then consider the non-discrete case analogous to Conjecture 1.

In decision problems with uncertainty, we maximize expected utility using a utility function instead of maximizing a value function. It is straightforward to show that utility is given by

$$U(z_1, z_2, \ldots, z_m) = \sum_{i=1}^{m} a_i u(z_i) \qquad \text{(A-8)}$$

if and only if additive independence is satisfied, and conditions analogous to (a)-(d) and (f) in Appendix B are met (Fishburn 1965). See Keeney and Raiffa (1976, Sections 6.5-6.6, pp. 295-307) and Kirkwood (1997, pp. 249-50) for further background, including assessment procedures for utility functions.

Analogously to Conjecture 1, we conjecture that it is possible to extend (A-8) to situations where the single attribute Z is defined at any location (x, y) within the region of interest. The corresponding utility function is given by:

$$U(z) = \iint_A a(x, y)u[z(x, y)]\,dxdy \tag{A-9}$$

if and only if additive independence, and conditions analogous to (a')-(d') and (f') in Appendix C are met. With these conditions, a plausibility argument for equation (A-9) can be obtained using reasoning analogous to the discussion in Appendix C. However, as in Conjecture 1, this plausibility argument would not constitute a proof. Results analogous to Theorem 2 and Conjecture 2, with a multiple attribute utility function and analogous assumptions can also be specified.

The required preference assumptions for an additive utility function are strong, however, and may not be appropriate in some decision situations. One possible approach to developing more general utility function forms with less restrictive requirements would be to construct the utility function over the value functions that were developed in Section 3 using methods such as those presented by Dyer and Sarin (1982) and Matheson and Abbas (2005). If the conditions needed for a value function of the form given in (1) hold, then in the case with discrete subregions, a utility function U could be constructed over the value function in (1) with the form:

$$U\left(V(z_1, z_2, \ldots, z_m)\right) = U\left(\sum_{i=1}^{m} a_i v(z_i)\right). \tag{A-10}$$

Similarly, in the non-discrete case, the utility function could be constructed over the value function in (3) with the form:

$$U\left(V(z)\right) = U\left(\iint_A a(x, y)v[z(x, y)]\,dxdy\right). \tag{A-11}$$

These are less restrictive than (A-8) and (A-9) in that they have unspecified utility functions U, and hence require less restrictive preference conditions than (A-8) or (A-9). Standard utility function assessment procedures can be modified to determine U. For example, a possible approach for assessing this utility function is to identify the potential decision consequences with the highest and lowest possible values, and visualize a hypothetical binary gamble between them with probability *p* of the highest-value consequence occurring and probability 1-*p* of the lowest-value consequence occurring. The utility of the value placed on a specified consequence could then be determined by finding the value of *p* for which the decision maker is indifferent between the specified consequence and the gamble. By equating expected utilities, the assessed *p* would be the utility associated with the specified consequence.

Additional References

Apostol, T. M. 1962. *Calculus, Volume II: Calculus of Several Variables with Applications to Probability and Vector Analysis*, Blaisdell Publishing Company, New York.

Debreu, G. 1960. Topological Methods in Cardinal Utility Theory. In *Mathematical Methods in the Social Sciences, 1959*, K. J. Arrow, S. Karlin, and P. Suppes (eds.), Stanford University Press, Stanford, CA, 1960, pp. 16-26. Downloaded from http://cowles.econ.yale.edu/P/cp/p01b/p0156.pdf on Nov. 29, 2010.

Dyer, J. S., R. K. Sarin. 1982. Relative Risk Aversion. *Management Science* **28**(8) 875-886.

Harvey, C. M., L. P. Østerdal, 2011. Integral Utility Models for Outcomes over Continuous Time. Working paper. Cited with permission of the authors.

Matheson, J. E., A. E. Abbas. 2005. Utility transversality: A value-based approach. *Journal of Multi-Criteria Decision Analysis* **13**(5-6) 229-238.

Small, C. G. 2007. *Functional Equations and How to Solve Them*, Springer Science+Business Media, New York.

Trench, W. F. 2003. *Introduction to Real Analysis.* Pearson Education. Upper Saddle River, NJ. Free Edition 1.05 downloaded on 11-29-10 from http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_REAL_ANALYSIS.PDF.