Automorphisms of Models of Arithmetic: a Unified View

Ali Enayat
Department of Mathematics and Statistics
American University
4400 Mass. Ave. N.W.
Washington, DC 20016-8050
e-mail: enayat@american.edu

May 17, 2006

Abstract

We develop the method of *iterated ultrapower representation* to provide a unified and perspicuous approach for building automorphisms of countable recursively saturated models of Peano arithmetic PA. In particular, we use this method to prove Theorem A below, which confirms a long standing conjecture of James Schmerl.

Theorem A. *If* $\mathfrak{M}$ *is a countable recursively saturated model of* PA *in which* $\mathbb{N}$ *is a strong cut, then for any* $\mathfrak{M}_0 \prec \mathfrak{M}$ *there is an automorphism* j *of* $\mathfrak{M}$ *such that the fixed point set of* j *is isomorphic to* $\mathfrak{M}_0$.

We also fine-tune a number of classical results. One of our typical results in this direction is Theorem B below, which generalizes a theorem of Kaye-Kossak-Kotlarski (in what follows $Aut(X)$ is the automorphism group of the structure X, and $\mathbb{Q}$ is the ordered set of rationals).

Theorem B. *Suppose* $\mathfrak{M}$ *is a countable recursively saturated model of* PA *in which* $\mathbb{N}$ *is a strong cut. There is a group embedding* $j \mapsto \hat{j}$ *from* $Aut(\mathbb{Q})$ *into* $Aut(\mathfrak{M})$ *such that for each* $j \in Aut(\mathbb{Q})$ *that is fixed point free,* $\hat{j}$ *moves every undefinable element of* $\mathfrak{M}$.

* *

2000 *Mathematics Subject Classification.* Primary 03F30, 03C62, 03H15; Secondary 03C15.

TABLE OF CONTENTS

* *

1. INTRODUCTION

This paper concerns automorphisms of countable recursively saturated models of PA (Peano arithmetic), an area of study that links logic, arithmetic, and permutation group theory in fascinating ways. In his original 1978 work on recursive saturation, Schlipf [Schl] had already noticed that every countable recursively saturated model has continuum many automorphisms, but the explicit study of automorphisms of countable recursively saturated models of arithmetic began with the papers of Smoryński [Sm-1] and Kotlarski [Kot-1] in the early 1980's, and has since grown into a fertile area of research (see the survey article [Kot-2] and the volume [KM] for a summary of progress up to the mid-1990's). The following theorem summarizes some of the striking results in the area that are relevant to this paper. Note that the converses to parts (a), (c), and (d) are known to be true. In what follows, $Aut(X)$ is the group of automorphisms of the structure X, and $\mathbb{Q}$ is the ordered set of rationals.

Theorem 1.1. *Suppose* $\mathfrak{M}$ *is a countable recursively saturated model of* PA.

(a) (Smoryński [Sm-1]) *If a cut* I *of* $\mathfrak{M}$ *is closed under exponentiation, then for some* $j \in Aut(\mathfrak{M})$, I *is the longest initial segment of* $\mathfrak{M}$ *that is pointwise fixed by* j.

(b) (Schmerl [Schm-1]) *There is a group embedding from* $Aut(\mathbb{Q})$ *into* $Aut(\mathfrak{M})$.

(c) (Kaye-Kossak-Kotlarski [KKK]) *If* $\mathbb{N}$ *is a strong cut of* $\mathfrak{M}$, *then there is some* $j \in Aut(\mathfrak{M})$ *such that every undefinable element of* $\mathfrak{M}$ *is moved by* j.

(d) (Kaye-Kossak-Kotlarski [KKK]) *If* $I \prec_{e,strong} \mathfrak{M}$, *then* I *is the fixed point set of some* $j \in Aut(\mathfrak{M})$.

(e) (Kossak [Kos-2]) *The number of isomorphism types of fixed point sets of* $\mathfrak{M}$ *is either* $2^{\aleph_0}$ *or* 1, *depending on whether* $\mathbb{N}$ *is a strong cut of* $\mathfrak{M}$, *or not.*

The *back-and-forth method* has been practically the only method for constructing interesting automorphisms of *countable recursively saturated models* of PA. Recently the author devised the method of *iterated ultrapower representation*[1] to establish the following result which combines parts (a) and (b) of Theorem 1.1.

Theorem 1.2. [E-2, Theorem B] *Suppose* $\mathfrak{M}$ *is a countable recursively saturated model of* PA *and* I *is a cut of* $\mathfrak{M}$ *that is closed under exponentiation. There is a group embedding*

$$j \longmapsto \hat{\jmath}$$

from $Aut(\mathbb{Q})$ *into* $Aut(\mathfrak{M})$ *such that for every nontrivial* $j \in Aut(\mathbb{Q})$ *the longest initial segment of* $\mathfrak{M}$ *that is pointwise fixed by* $\hat{\jmath}$ *is* I. *Moreover, for every fixed point free* $j \in Aut(\mathbb{Q})$, *the fixed point set of* $\hat{\jmath}$ *is isomorphic to* $\mathfrak{M}$.

In this paper we provide further evidence of the versatility of the methodology of iterated ultrapower representation by using this method to establish a long standing conjecture of Schmerl, and to provide refinements of several known results, including parts (c), (d), and (e) of Theorem 1.1. We should emphasize that the specific machinery of iterated ultrapowers developed in

[1] Schmerl also used this method to prove the existence of an automorphism of an arithmetically saturated model $\mathfrak{M}$ of $Th(\mathbb{N})$ such that $j(a) > a$ for all nonstandard elements a of $\mathfrak{M}$, see [Kos-3, Question 4.5].

this paper is a generalization of Gaifman's technology of *minimal types* in [Ga][2].

The paper is organized as follows. Section 2 is devoted to preliminary results and definitions. Section 3 is the central technical section of the paper in which the general machinery of iterated ultrapowers of models of arithmetic is developed. Section 4 concentrates on the consequences of Section 3 for countable recursively saturated models of PA. In particular, in Section 4.2 we establish a conjecture of Schmerl by showing that the isomorphism type of any elementary submodel of a countable arithmetically saturated model $\mathfrak{M}$ of PA can be realized as the isomorphism type of the fixed point set of some automorphism of $\mathfrak{M}$. Section 4 also includes new proofs of (generalizations of) parts (b) through (e) of Theorem 1.1 (see 4.1, 4.3, and 4.4). Finally, in Section 4.5 we unify parts (c) and (d) of Theorem 1.1 by establishing a general theorem that shows, as a corollary, that if I is a strong cut of a countable recursively saturated $\mathfrak{M}$ of PA, and $\mathfrak{M}_0$ is an elementary submodel of $\mathfrak{M}$ such that for some $c \in M$, $M_0 := \{(c)_i : n \in I\}$, then M_0 can be realized as the fixed point set of some automorphism of $\mathfrak{M}$ (here $(c)_i$ = the exponent of the i-th prime in the prime factorization of c). This latter result was announced in [KKK, Theorem 5.7] (and attributed to Schmerl), but the proof was never published.

Acknowledgments: I am grateful to Roman Kossak and Jim Schmerl for sharing their knowledge of models of arithmetic with me throughout the past two decades. I am also indebted to the anonymous referee for suggesting a number of helpful editorial and mathematical improvements (as noted throughout the paper).

2. PRELIMINARIES

In this section we review definitions, conventions, and known results that will be utilized in this paper.

- Our language of first order arithmetic, $\mathcal{L}_A$, is $\{+, \cdot\}$. For a language $\mathcal{L}$ extending $\mathcal{L}_A$, $PA(\mathcal{L})$ is PA augmented by the induction scheme for

[2]Minimal types provide a robust method for constructing various models of PA. In particular, as shown in [Ga], they can be employed to build models of PA whose automorphism group is isomorphic to the automorphism group of a prescribed linear order (this refines the classical work of Ehrenhfeucht and Mostowski in the context of arithmetic). See Schmerl's [Schm-2] for a recent generalization of Gaifman's aforementioned result.

all $\mathcal{L}$-formulas. When $\mathcal{L}$ is clear from the context, we shall follow a common practice from the literature and use PA^* to refer to $PA(\mathcal{L})$.

- μ is Kleene's least search operator. More specifically, $x = \mu t\ \varphi(t)$ if x is the least solution of $\varphi(x)$, with the understanding that $\mu t\ \varphi(t) = 0$ if $\exists t\ \varphi(t)$ fails.

- We often use Gothic fonts to denote models, and the corresponding Roman fonts to denote their universes, e.g., a model of PA is of the form $\mathfrak{M} := (M, \oplus^{\mathfrak{M}}, \otimes^{\mathfrak{M}})$.

- Every model of PA has an initial segment $\mathbb{N}$ consisting of the standard elements of $\mathfrak{M}$. We use $\mathbb{N}^+$ to denote $\mathbb{N}\backslash\{0\}$.

- Given a model $\mathfrak{M}$ of $PA(\mathcal{L})$, a subset S of M is said to be *parametrically definable in* $\mathfrak{M}$ (abbreviated: $\mathfrak{M}$-*definable*) if there is some formula $\varphi(x, u)$ of $\mathcal{L}$, and some parameter $a \in M$, such that S is the solution set of $\varphi(x, a)$, i.e.,

$$S = \{m \in M : (\mathfrak{M}, a) \models \varphi(m, a)\}.$$

- ACA_0 is the well-known subsystem of second order arithmetic with the comprehension scheme limited to formulas with no second order quantifiers, as in [Si]. Models of ACA_0 are of the *two-sorted* form $(\mathfrak{M}, \mathcal{A})$, where $\mathfrak{M}$ is a model of PA, and $\mathcal{A}$ is a family of subsets of M. Note that if $(\mathfrak{M}, \mathcal{A}) \models ACA_0$, then $(\mathfrak{M}, S)_{S\in\mathcal{A}} \models PA^*$ and conversely, if $(\mathfrak{M}, S)_{S\in\mathcal{A}} \models PA^*$ and $\mathcal{A}$ is the family of subsets of M that are parametrically definable in $(\mathfrak{M}, S)_{S\in\mathcal{A}}$, then $(\mathfrak{M}, \mathcal{A}) \models ACA_0$.

- For models $\mathfrak{M}$ and $\mathfrak{N}$ of PA, $\mathfrak{N}$ *end extends* $\mathfrak{M}$ (equivalently: $\mathfrak{M}$ is an *initial* submodel of $\mathfrak{N}$), written $\mathfrak{M} \subseteq_e \mathfrak{N}$, if $\mathfrak{M}$ is a submodel of $\mathfrak{N}$ and $a < b$ for every $a \in M$, and $b \in N\backslash M$.

- We write $\mathfrak{M} \prec_e \mathfrak{N}$ if $\mathfrak{N}$ is an elementary end extension of $\mathfrak{M}$; and we write $\mathfrak{M} \prec_c \mathfrak{N}$ if $\mathfrak{N}$ is a *cofinal* elementary extension of $\mathfrak{N}$.

- For $j \in Aut(\mathfrak{M})$, $\mathit{fix}(j) := \{m \in M : j(m) = m\}$. Since models of PA are equipped with definable Skolem functions, for a model $\mathfrak{M} \models PA$, $\mathit{fix}(j)$ is the universe of an *elementary* submodel of $\mathfrak{M}$.

- There is a first order formula $E(x, y)$ in the language of arithmetic that expresses "the x-th digit in the binary expansion of y is 1". $E(x, y)$ in many ways behaves like the membership relation, indeed it is well-known that if $\mathfrak{M}$ is a model of PA, then (M, E) is a model of $ZF \backslash \{\text{Infinity}\} \cup \{\neg\text{Infinity}\}$. We shall henceforth refer to E as "*Ackermann's* $\in$". Ackermann's $\in$ allows us to simulate finite set theory and combinatorics within models of arithmetic.

- A subset X of M is *coded* in $\mathfrak{M}$ if for some $c \in M$,

  $$X = c_E := \{x \in M : \mathfrak{M} \models xEc\}.$$

  It is well-known that a subset X of a model $\mathfrak{M}$ of PA is coded in $\mathfrak{M}$ iff X is bounded and $\mathfrak{M}$-definable.

- Given $m \in M$, $\overline{m} := \{x \in M : x < m\}$. Note that $\overline{m}$ is coded in $\mathfrak{M}$ even when m is nonstandard.

- A *cut* I of $\mathfrak{M}$ is an initial segment of $\mathfrak{M}$ with no last element (N.B. in this paper $I = M$ is allowed). When a cut I is closed under multiplication (and therefore under addition as well), we shall use $\mathbf{I}$ to refer to the submodel of $\mathfrak{M}$ induced by I, i.e.,

  $$\mathbf{I} := (I, \oplus^{\mathfrak{M}}, \otimes^{\mathfrak{M}}).$$

- The *overspill principle* for models $\mathfrak{M}$ of PA states that if the $\mathfrak{M}$-solution set S of some first order unary formula $\varphi(x)$ (with suppressed parameters) includes a proper cut I of $\mathfrak{M}$, then for some $a \in M \backslash I$, $\overline{a} \subseteq S$.

- For a cut I of $\mathfrak{M}$, $SSy_I(\mathfrak{M})$ is the family consisting of sets of the form $S \cap I$, where S is an $\mathfrak{M}$-definable subset of M. In particular, if $I = M$, then $SSy_I(\mathfrak{M})$ is simply the collection of $\mathfrak{M}$-definable subsets of M. It is well-known that if I is a proper initial segment of a model $\mathfrak{M}$ of PA, then

  $$SSy_I(\mathfrak{M}) = \{c_E \cap I : c \in M\}.$$

  When $I = \mathbb{N}$, we shall use the commonly used notation $SSy(\mathfrak{M})$ instead of $SSy_{\mathbb{N}}(\mathfrak{M})$.

- I is a *strong cut* of $\mathfrak{M}$ if, for each function f whose graph is coded in $\mathfrak{M}$ and whose domain includes I, there is some s in M such that for all $m \in M$, $f(m) \notin I$ iff $s < f(m)$. Note that $\mathfrak{M}$ is considered to be a strong cut of itself. Kirby and Paris proved that strong cuts of models of PA are themselves models of PA [KP, Proposition 8]. Indeed, their proof shows the following more general result (which appears in [Ki-1], see also [E-1, Lemma A.4]).

Theorem 2.1. (Kirby-Paris) *The following are equivalent for a cut* I *of* $\mathfrak{M} \models PA$ *:*
(a) I *is strong in* $\mathfrak{M}$.
(b) $(\mathbf{I}, SSy_I(\mathfrak{M})) \models ACA_0$.

- Let $(\mathfrak{M},\mathcal{A}) \models ACA_0$. We say that a model $\mathfrak{A} = (A, \oplus^{\mathfrak{A}}, \otimes^{\mathfrak{A}})$ of PA is *interpretable* in $(\mathfrak{M},\mathcal{A})$ if $A = M$, and the operations of $\mathfrak{A}$ are coded in $\mathcal{A}$, i.e., the sets

$$\{\langle x, y, z\rangle : x \oplus^{\mathfrak{A}} y = z,\ (x, y, z) \in M^3\}$$

and

$$\{\langle x, y, z\rangle : x \otimes^{\mathfrak{A}} y = z,\ (x, y, z) \in M^3\}$$

are both members of $\mathcal{A}$. Here $\langle x, y, z\rangle \in M$ is the canonical code for the ordered triple (x, y, z).

- Suppose $\mathfrak{A}$ is interpretable in $(\mathfrak{M}, \mathcal{A}) \models ACA_0$ and let $F \subseteq M$ be the set of Gödel numbers $\ulcorner\varphi\urcorner$ of $\mathcal{L}_A$-formulas φ, as computed in $\mathfrak{M}$ (note that F will include nonstandard elements if $\mathfrak{M}$ is nonstandard). An element S of $\mathcal{A}$ is said to be a *satisfaction predicate* for $\mathfrak{A}$ if S correctly codes the satisfaction relation of $\mathfrak{A}$ for (at least) all standard elements of F. This can be more explicitly described by saying that S satisfies the axiom scheme consisting of the sentence (1) plus the collection of sentences (2_n) below (for each standard n), all formulated in the language of arithmetic augmented with a predicate S:

  (1) S consists of coded ordered pairs of the form $\langle\ulcorner\varphi\urcorner, a\rangle$ (intuitively speaking $\langle\ulcorner\varphi(x)\urcorner, a\rangle \in S$ expresses "$\varphi(a)$ is true");

  (2_n) [S is n-correct] S satisfies Tarski's inductive conditions for a truth predicate for all formulas of quantifier rank n (including any nonstandard ones).

- If $(\mathfrak{M},S) \models PA(S)$, and S is a satisfaction predicate for $\mathfrak{M}$, then we say that S is *satisfaction class*[3] for $\mathfrak{M}$. It is well-known that for each fixed standard n_0 there is parameter free definable subset S_{n_0} of $\mathfrak{M}$ that satisfies conditions (1) and (2_{n_0}) above, but of course Tarski's undefinability of truth theorem dictates that any $\mathfrak{M}$-definable S that satisfies (1) must fail (2_n) for some n. Note that if S is a satisfaction

[3]Satisfaction classes in this sense are referred to as *partial inductive satisfaction classes* in [Ka].

class of a nonstandard model of PA, then S is s-correct for some *nonstandard* s. This follows from overspill and the fact that (2_n) can be uniformly expressed by a single formula with parameter n in the language of arithmetic augmented with a predicate S.

- $\mathfrak{M}$ is *recursively saturated* if for every finite sequence $\mathbf{m}$ of elements of M, every recursive finitely realizable type over the expanded model $(\mathfrak{M}, \mathbf{m})$ is realized in $\mathfrak{M}$.

The concepts of recursive saturation and satisfaction predicates and classes are intimately tied, as witnessed by the following result.

Theorem 2.2.
(a) (Folklore) *If* $\mathfrak{A}$ *is interpretable in a nonstandard model* $(\mathfrak{M},\mathcal{A})$ *of* ACA_0 *and some* $S \in \mathcal{A}$ *is a satisfaction predicate for* $\mathfrak{A}$*, then* $\mathfrak{A}$ *is recursively saturated.*
(b) (Barwise-Schlipf [BS]) *A countable nonstandard model* $\mathfrak{M}$ *of* PA *is recursively saturated iff* $\mathfrak{M}$ *has a satisfaction class.*

The following two isomorphism theorems play a central role in the proofs of the principal results of Section 4.

Theorem 2.3. (Folklore[4]) *The isomorphism type of a countable recursively saturated model* $\mathfrak{M}$ *of arithmetic is uniquely determined by the two invariants* $Th(\mathfrak{M})$ *and* $SSy(\mathfrak{M})$.

In order to state the next isomorphism theorem we need to recall the following definition:

- Suppose I is a proper cut of $\mathfrak{M}$. I is $\mathbb{N}$-*coded from above* in $\mathfrak{M}$ iff for some $c \in M$, $\{(c)_n : n \in \mathbb{N}\}$ is downward cofinal in $M \backslash I$ (here $(c)_n$ is the exponent of the n-th prime in the prime factorization of c within $\mathfrak{M}$). It is easy to see that if I is a strong cut of $\mathfrak{M}$, then I is not $\mathbb{N}$-coded from above in $\mathfrak{M}$ (but the converse is false).

Theorem 2.4. (Kossak-Kotlarski [KK, Theorem 2.1, Corollary 2.3]) *Suppose* $\mathfrak{M}$ *and* $\mathfrak{M}^*$ *are countable recursively saturated models of* PA*, and* I *is a cut shared by both* $\mathfrak{M}$ *and* $\mathfrak{M}^*$ *which satisfies the following two properties:*
(a) I *is* $\mathbb{N}$*-coded from above in neither* $\mathfrak{M}$ *nor* $\mathfrak{M}^*$*, and*

[4] The origins of this result go back to the important paper of Jensen and Ehrenfeucht [JE], whose work shows that this result holds not only for PA, but also for the class of all "rich" theories, a class that also includes ZF and RCF (the theory of real closed fields). This is elaborated in Smorynski's survey article [Sm-2].

(b) $SSy_I(\mathfrak{M}) = SSy_I(\mathfrak{M}^*)$.
Then $\mathfrak{M}$ *and* $\mathfrak{M}^*$ *are isomorphic over* I *(i.e., there is an isomorphism between* $\mathfrak{M}$ *and* $\mathfrak{M}^*$ *that is the identity on* I*).*

Finally, we consider the following strengthening of recursively saturated models that has come to play a key role in the model theory of PA (see [KS-1]).

- A model $\mathfrak{M}$ is *arithmetically saturated* if for every finite sequence $\mathbf{m}$ of elements of M, every finitely realizable type over the expanded model $(\mathfrak{M}, \mathbf{m})$ that is arithmetical in the type of $\mathbf{m}$ is realized in $\mathfrak{M}$.

Note that a usual compactness argument shows that every consistent extension of PA has a countable arithmetically saturated model. The following result shows that arithmetical saturation is far more natural than what its definition suggests at first sight (the countability requirement is not needed to establish $(c) \Rightarrow (a) \Leftrightarrow (b)$).

Theorem 2.5. (Kaye-Kossak-Kotlarski [KKK]) *The following are equivalent for a countable recursively saturated model* $\mathfrak{M}$ *of* PA*:*
(a) $\mathfrak{M}$ *is arithmetically saturated.*
(b) $\mathbb{N}$ *is a strong cut of* $\mathfrak{M}$*.*
(c) *There is an automorphism of* $\mathfrak{M}$ *that moves every undefinable element of* $\mathfrak{M}$*.*

3. ITERATED ULTRAPOWERS OF MODELS OF ARITHMETIC IN A GENERAL SETTING

This section develops the general framework for the iterated ultrapowers that are employed in this paper. In contrast with the usual construction of ultrapowers in general model theory where *all* functions from some index set I into the universe M of a model $\mathfrak{M}$ are used in the formation of the ultrapower, model theorists of arithmetic have found it useful to consider "limited" ultrapowers in which a manageable family of functions from I to M are selected to craft the ultrapower. The following three varieties (a), (b), and (c) of limited ultrapowers are the most well-known in the model theory of arithmetic:

(a) *Skolem-Gaifman ultrapowers*, where the index set I is identical to the universe M of the model $\mathfrak{M}$, and the family of functions used in the formation of the ultrapower is the set of all $\mathfrak{M}$-definable ones. This sort of ultrapower was implicitly used by Skolem in his original construction of a nonstandard model of arithmetic. Later, they were explicitly employed by MacDowell and Specker in the proof of their celebrated theorem [MS]. Ultimately, in Gaifman's hands [Ga] they were refined to a high degree of sophistication to produce a variety of striking results. One of Gaifman's fundamental discoveries was that the ultrapower construction can be iterated along *any* linear order with appropriately chosen ultrafilters.

(b) *Kirby-Paris ultrapowers,* where the index is a proper cut I of $\mathfrak{M}$, and the family of functions used in the formation of the ultrapower are functions f such that for some function g coded in $\mathfrak{M}$, $f = g \upharpoonright I$. This brand of ultrapower first appeared in [KP] and has proved to a valuable tool in the study of cuts of nonstandard models of arithmetic. Clote [C] employed finite iterations of this type of ultrapower to answer a question of Mills and Paris regarding the relationship between n-Ramsey and n-extendible cuts.

(c) *Paris-Mills ultrapowers,* where the index set is of the form $\overline{m}$ for some nonstandard m in $\mathfrak{M}$, and the functions used are those that are coded in $\mathfrak{M}$. This type of ultrapower was first considered by Paris and Mills in [PM] to show, among other things, that one can arrange a model of PA in which there is an externally countable nonstandard integer H such that the external cardinality of $Superexp(2, H)$ is of any prescribed infinite cardinality. Here $Superexp(x, y)$ is the result of y iterations of the exponential function 2^x. Paris-Mills ultrapowers were used in [E-2] to establish Theorem 1.2.

In Sections 4.1 and 4.4 we shall employ iterations of Skolem-Gaifman ultrapowers à la Gaifman, but in Sections 4.2, 4.3 and 4.5 we need to iterate a different variety of ultrapowers that happen to be a generalization of both Skolem-Gaifman and Kirby-Paris ultrapowers. We shall now set up a sufficiently general framework for handling the machinery involved in all cases under consideration in this paper[5]. Our work was inspired by - and generalizes - Gaifman's technology of minimal types developed in [Ga].

3.1. Functions

In what follows I is a cut of a model $\mathfrak{M}$ of $PA(\mathcal{L})$ that is closed under multiplication.

[5] Iterated Paris-Mills ultrapowers have a great deal of similarity to the iterated ultrapowers in this paper, but strictly speaking, they do not fall into the framework developed here. A key difference between these two types of ultrapowers is that the cuts fixed by the Paris-Mills ultrapower only need to be closed under exponentiation, but those fixed by the ultrapowers in this paper need to satisfy the far more stringent condition of being a model of full PA.

- Suppose X and Y are subsets of M, Y^X is the set of all functions from X to Y and $(Y^X)^{\mathfrak{M}}$ is the collection of all $f \in Y^X$ such that $f = g \cap (X \times Y)$ for some g that is $\mathfrak{M}$-definable.

- For $\mathcal{F} \subseteq M^I$, $B(\mathcal{F}) := \{S \subseteq I : \chi_S \in \mathcal{F}\}$, where χ_S is the characteristic function of S (in all the cases under consideration here $B(\mathcal{F})$ will be a Boolean algebra). Given $n \in \mathbb{N}^+$, $\mathcal{F}$ also gives rise to the family of functions $\mathcal{F}_n \subseteq M^{(I^n)}$ that are canonically coded in $\mathcal{F}$. More specifically,

  $$\mathcal{F}_n := \{f_n : f \in \mathcal{F}\},$$

  where $f_n(x_1, \cdots, x_n) := f(\langle x_1, \cdots, x_n\rangle)$, and $(x_1, \cdots, x_n) \longmapsto \langle x_1, \cdots, x_n\rangle$ is a *canonical bijection* between M^n and M. $\mathcal{F}_n$ in turn gives rise to

  $$B(\mathcal{F}_n) := \{S \subseteq I^n : \chi_S \in \mathcal{F}_n\}.$$

  Note that we identify $\mathcal{F}_1$ with $\mathcal{F}$, and therefore $B(\mathcal{F}_1) = B(\mathcal{F})$.

- A family $\mathcal{F} \subseteq M^I$ is said to be $\mathfrak{M}$-*adequate* if $\mathcal{F}$ satisfies the following properties (1) - (3):

  (1) [*Amenability property*] $(\mathbf{I}, B(\mathcal{F})) \models ACA_0$. In particular, $\mathbf{I}$ is a model of PA.

  (2) [*Constant property*] For each $m \in M$, $\mathcal{F}$ contains the constant function $c_m : I \to \{m\}$.

  (3) [*Skolem property*] For every formula $\varphi(x_1, \cdots, x_n, y)$ of $\mathcal{L}$ and every $f_1, \cdots, f_n$ in $\mathcal{F}$, there is some $g \in \mathcal{F}$ such that for all $i \in I$

  $$\mathfrak{M} \models (\exists y\ \varphi(f_1(i), \cdots, f_n(i), y) \to \varphi(f_1(i), \cdots, f_n(i), g(i)))\,.$$

Proposition 3.1.1[6]**.** *If $\mathcal{F} \subseteq M^I$ satisfies the Skolem property and $n \in \mathbb{N}^+$, then:*
(a) *$B(\mathcal{F}_n)$ is a Boolean algebra.*
(b) [*Measurability property*] *For all formulas $\varphi(x_1, \cdots, x_k)$ of $\mathcal{L}$, and all functions $f_1, \cdots, f_k$ from $\mathcal{F}_n$,*

$$\{(i_1, \cdots, i_n) \in I^n : \mathfrak{M} \models \varphi(f_1(i_1, \cdots, i_n), \cdots, f_k(i_1, \cdots, i_n))\} \in B(\mathcal{F}_n).$$

[6] We are grateful to the referee for suggesting this proposition. In the first draft of this paper, the measurability property was built into the definition of $\mathfrak{M}$-adequacy, and the closure of $B(\mathcal{F})$ under Boolean operations was derived from the amenability property.

(c) $\mathcal{F}_n$ *has the Skolem property, i.e., for every formula* $\varphi(x_1, \cdots, x_k, y)$ *of* $\mathcal{L}$ *and every* $f_1, \cdots, f_k$ *in* $\mathcal{F}_n$*, there is some* $g \in \mathcal{F}_n$ *such that for all* $\mathbf{i} := (i_1, \cdots, i_n) \in I^n$

$$\mathfrak{M} \models \exists y\ \varphi(f_1(\mathbf{i}), \cdots, f_n(\mathbf{i}), y) \to \varphi(f_1(\mathbf{i}), \cdots, f_n(\mathbf{i}), g(\mathbf{i})).$$

Proof: Thanks to the existence of a canonical bijection between I and I^n, all three parts of the proposition hinge on the special case of $n = 1$. In this light, we shall only establish (a) and (b) for $n = 1$.

(a) Suppose f_1 and f_2 are the characteristic functions of S_1 and S_2 in $B(\mathcal{F})$. To verify the closure of $B(\mathcal{F})$ under intersections, consider the formula

$$\theta(x_1, x_2, y) := (x_1 = x_2 = y = 1).$$

Note that $i \in S_1 \cap S_2$ iff $\mathfrak{M} \models \exists y \theta(f_1(i), f_2(i), y)$. Therefore the Skolem function g for θ is the characteristic function of $S_1 \cap S_2$. Similarly, to see that $B(\mathcal{F})$ is closed under complementation, consider the formula

$$\psi(x, y) := (x \neq y) \wedge (y = 0 \vee y = 1).$$

Then $\forall i \in I\ \mathfrak{M} \models \exists y \psi(f_1(i), y)$, and the Skolem function for this formula is the characteristic function of the complement of S_1.

(b) Given a formula $\varphi(x_1, \cdots, x_k)$, let $S = \{i \in I : \mathfrak{M} \models \varphi(f_1(i), \cdots, f_k(i))\}$ and consider the formula

$$\delta(x_1, \cdots, x_k, y) := ((y = 1) \leftrightarrow \varphi(x_1, \cdots, x_k)) \wedge (y = 0 \vee y = 1).$$

The Skolem function for δ will be the characteristic function of either S or $I \backslash S$. Since $B(\mathcal{F})$ is closed under complementation, this completes the proof. $\square$

If $\mathcal{F}$ is the collection $M^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to M, then clearly $\mathcal{F}$ is $\mathfrak{M}$-adequate. The following proposition provides the $\mathfrak{M}$-adequate families that will be utilized in this paper.

Proposition 3.1.2. $\mathcal{F} := \left(M^I\right)^{\mathfrak{N}}$ *is* $\mathfrak{M}$*-adequate if* $\mathfrak{M} \models PA^*$, $\mathfrak{M} \preceq \mathfrak{N}$, I *is a cut of both* $\mathfrak{M}$ *and* $\mathfrak{N}$, *and* I *is strong in* $\mathfrak{N}$ *(N.B., I need not be strong in* $\mathfrak{M}$*).*

Proof: If $I = M = N$, then $\mathcal{F}$ is the family of parametrically $\mathfrak{M}$-definable functions from M to M, where the properties (1) - (3) are well-known and easy to verify, so we concentrate on the case $I \subsetneq N$.

(1) Amenability is a direct consequence of Theorem 2.1.

(2) The constant property is obvious.

(3) The Skolem property hinges on the first order definability of the μ-operator in models of arithmetic. More specifically, given $\varphi(x_1, \cdots, x_n, y)$ of $\mathcal{L}$ and $f_1, \cdots, f_n$ in $\mathcal{F}$, choose $\widetilde{f}_1, \cdots, \widetilde{f}_n$ in N such that $\widetilde{f}_i \upharpoonright I = f_i$ for $1 \leq i \leq n$, and consider the function $\widetilde{g}(x)$ defined in $\mathfrak{N}$ as

$$\widetilde{g}(x) := \mu y\ \varphi(\widetilde{f}_1(x), \cdots, \widetilde{f}_n(x), y).$$

Clearly $g := \widetilde{g} \upharpoonright I$ is the desired Skolem function.

□

Remark 3.1.3.

(a) As we shall see, Proposition 3.1.2 has a converse when $\mathcal{F}$ is countable. More specifically, By Proposition 3.2.1 and part (b) of Theorem 3.3.6, if $\mathcal{F}$ is a countable $\mathfrak{M}$-adequate family of functions, there is an elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ such that I is a strong cut of $\mathfrak{N}$ and $\mathcal{F} := \left(M^I\right)^{\mathfrak{N}}$. As pointed out by the referee, this provides an alternative definition of $\mathfrak{M}$-adequacy for countable $\mathcal{F}$.

(b) The simplest family of $\mathfrak{M}$-adequate functions given by Proposition 3.1.2 corresponds to the case $I = M = N$, where $\left(M^I\right)^{\mathfrak{N}}$ yields the family of functions used in Skolem-Gaifman ultrapowers. On the other hand, the family of functions used in Kirby-Paris ultrapowers corresponds to the case where $I \subsetneq M = N$. The third family of functions corresponds to the case $I \subsetneq M \subsetneq N$, which does not seem to have been used before in the literature. This new family of functions plays a key role in the proof of Schmerl's conjecture (Theorem 4.2.1) and Theorem 4.5.1.

3.2. Ultrafilters

In order to build and analyze iterated ultrapowers of $\mathfrak{M}$ using $\mathfrak{M}$-adequate families $\mathcal{F}$, we first need to focus on ultrafilters over Boolean algebras of the form $B(\mathcal{F})$, and more generally $B(\mathcal{F}_n)$.

- In what follows, assume that $\mathfrak{M}$ is a model of $PA(\mathcal{L})$, I is a cut of $\mathfrak{M}$ that is closed under multiplication, $\mathcal{F}$ is an $\mathfrak{M}$-adequate family, and $n \in \mathbb{N}^+$.

- A subset $\mathcal{U}$ of $B(\mathcal{F}_n)$ is a *filter* if $\mathcal{U}$ is closed under intersections, and $\mathcal{U}$ is upward closed.

- A filter $\mathcal{U}$ on $B(\mathcal{F}_n)$ is $\mathcal{F}_n$*-complete* if for every $f \in \mathcal{F}_n$ and every $c \in I$ with $f : I^n \to \overline{c}$, there is some $X \in \mathcal{U}$ such that f is constant on X. It is easy to see that if $\mathcal{U}$ is $\mathcal{F}_n$-complete, then $\mathcal{U}$ is an *ultrafilter* on $B(\mathcal{F}_n)$ since for each $Y \in B(\mathcal{F}_n)$, the characteristic function of Y is constant on some member of $\mathcal{U}$.

- Every $f : I \to \{0,1\}$ in $\mathcal{F}$ gives rise to a sequence $\left\langle S^f_{n,i} : i \in I \right\rangle$ of members of $B(\mathcal{F}_n)$, where

$$S^f_{n,i} := \{(j_1, \cdots, j_n) \in I^n : f(\langle i, j_1, \cdots, j_n \rangle) = 1\}.$$

- A filter $\mathcal{U}$ on $B(\mathcal{F}_n)$ is $\mathcal{F}_n$*-definable*[7], if $\mathcal{U}$ is $\mathcal{F}_n$-complete, and for every $f : I \to \{0,1\}$ in $\mathcal{F}$

$$\{i \in I : S^f_{n,i} \in \mathcal{U}\} \in B(\mathcal{F}).$$

  Note that since we identify $\mathcal{F}_1$ with $\mathcal{F}$, a filter $\mathcal{U}$ on $B(\mathcal{F})$ is $\mathcal{F}$*-definable* if $\mathcal{U}$ is $\mathcal{F}_1$-definable.

- For a linearly ordered set X, $[X]^n$ is the collection of increasing n-tuples from X.

- Suppose $f : [I]^n \to M$. A subset X of I is f*-canonical* if there is some $S \subseteq \{1, \cdots, n\}$ such that for all sequences $s_1 < \cdots < s_n$, and $t_1 < \cdots < t_n$ of elements of X,

$$f(s_1, \cdots, s_n) = f(t_1, \cdots, t_n) \Longleftrightarrow \forall i \in S(s_i = t_i).$$

  Note that if $S = \emptyset$, then f is constant on $[X]^n$, and if $S = \{1, \cdots, n\}$, then f is injective on $[X]^n$.

- A filter $\mathcal{U}$ is $(\mathcal{F}, n)$*-canonically Ramsey* if for every $f : [I]^n \to M$ with $f \in \mathcal{F}_n$, there is some $X \in \mathcal{U}$ such that X is f-canonical. $\mathcal{U}$ is $\mathcal{F}$*-canonically Ramsey* if $\mathcal{U}$ is $(\mathcal{F}, n)$-canonically Ramsey for all $n \in \mathbb{N}^+$.

- A filter $\mathcal{U}$ on $B(\mathcal{F})$ is $(\mathcal{F}, n)$*-Ramsey,* if for every $f : [I]^n \to \{0,1\}$ with $f \in \mathcal{F}$, there is some $X \in \mathcal{U}$ which is f-homogeneous, i.e., $f \restriction [X]^n$ is constant. $\mathcal{U}$ is $\mathcal{F}$*-Ramsey* if $\mathcal{U}$ is $(\mathcal{F}, n)$-Ramsey for all $n \in \mathbb{N}^+$.

[7] This terminology was coined by Gaifman [Ga] in the equivalent context of types, see also [Ki-2]. Definable ultrafilters were dubbed "iterable" in [E-1], since they are amenable to iteration.

Theorem 3.2.1[8]**.** *Suppose I, $\mathfrak{M}$, $\mathfrak{N}$ and $\mathcal{F}$ are as in Proposition* 3.1.2*, and that N and $\mathcal{L}$ are both countable. Then $B(\mathcal{F})$ carries a nonprincipal ultrafilter $\mathcal{U}$ that is $\mathcal{F}$-definable, $\mathcal{F}$-Ramsey and $\mathcal{F}$-canonically Ramsey.*

Proof: As shown in [E-1, Section 3.2.1], the desired $\mathcal{U}$ can be chosen to be a "generic" ultrafilter, where the forcing conditions are members of $B(\mathcal{F})$ that are unbounded in I. The proof hinges on the fact that the relevant combinatorial density theorems can all be implemented in ACA_0 (this is where the amenability property and Theorem 2.1 come into play).
□

We now briefly analyze the relationship between various combinatorial properties of ultrafilters in our general setting. As we shall see, as soon as an ultrafilter $\mathcal{U}$ on $B(\mathcal{F})$ is $(\mathcal{F}, 2)$-Ramsey, then it will also be $\mathcal{F}$-definable and $\mathcal{F}$-canonically Ramsey. Let us introduce one more definition:

- $\mathcal{U}$ is *$\mathcal{F}$-minimal*[9] if for every $f \in \mathcal{F}$ there is some $X \in \mathcal{U}$ on which f is either constant or injective. Note that $\mathcal{U}$ is $\mathcal{F}$-minimal iff $\mathcal{U}$ is $(\mathcal{F}, 1)$-canonically Ramsey.

Proposition 3.2.2. *Suppose $\mathcal{F} \subseteq M^I$ is $\mathfrak{M}$-adequate, I is a strong cut of $\mathfrak{M}$, and $\mathcal{U}$ is an ultrafilter on $B(\mathcal{F})$. Then:*
(a) *For all $n \in \mathbb{N}^+$, $\mathcal{U}$ is $(\mathcal{F}, 2n)$-Ramsey $\Rightarrow \mathcal{U}$ is $(\mathcal{F}, n)$-canonically Ramsey,*
(b) *$\mathcal{U}$ is $(\mathcal{F}, 2)$-Ramsey $\Rightarrow \mathcal{U}$ is $\mathcal{F}$-definable,*
(c) *$\mathcal{U}$ is $(\mathcal{F}, 2)$-Ramsey $\Rightarrow \mathcal{U}$ is $\mathcal{F}$-minimal, and*
(d) *$\mathcal{U}$ is $\mathcal{F}$-definable and $\mathcal{F}$-minimal $\Rightarrow \mathcal{U}$ is $\mathcal{F}$-Ramsey.*

Proof (outline):

(a): This follows from implementing any of the proofs of the Erdős-Rado canonical partition theorem for exponent n from Ramsey's theorem for exponent $2n$ (see the original proof in [ER], the more transparent [R], or the recently discovered one by Mileti [Mi]).

(b): It is not hard to see that if $\mathcal{U}$ is $(\mathcal{F}, 3)$-Ramsey, then $\mathcal{U}$ is $\mathcal{F}$-definable ([Ki-2, Lemma 1.9]). Kirby [Ki-2, Theorem D] strengthened this result by

[8]The origins of this result are to be found in the work of MacDowell-Specker [MS], which only addresses the existence of $\mathcal{F}$-complete ultrafilters. However, the techniques in [MS] can be used to show that the countability restrictions in Theorem 3.2.1 can be lifted for the case $I = M = N$. This was first demonstrated by Gaifman [Ga].

[9]This terminology is related to the Rudin-Keisler ordering on ultrafilter: Rudin-Kesiler minimal ultrafilters are precisely those on which all the relevant functions are either constant or injective.

showing that the $(\mathcal{F},3)$-Ramsey property follows from the $(\mathcal{F},2)$-Ramsey property.

(c): Given $f : I \to I$ with $f \in \mathcal{F}$, color $\{i,j\} \in [I]^2$ red if $f(i) = f(j)$, and otherwise color $\{i,j\}$ blue. Then f will be constant or injective on a monochromatic set.

(d): This is based on the observation that Kunen's theorem establishing "minimal $\Rightarrow$ Ramsey" in the context of ultrafilters over $\mathcal{P}(\mathbb{N})$ (as in [J, Lemma 38.1]) generalizes to the present setting.
$\square$

Corollary 3.2.3. *Suppose $\mathcal{F}$ and $\mathcal{U}$ are as in Proposition* 3.2.2. *If $\mathcal{U}$ is $(\mathcal{F},2)$-Ramsey, then $\mathcal{U}$ is $\mathcal{F}$-Ramsey, $\mathcal{F}$-canonically Ramsey and $\mathcal{F}$-definable.*

Remark 3.2.4. Phillips [Ph] and later Potthoff [Po] showed that $\mathcal{F}$-minimality does not imply $\mathcal{F}$-definability when $I = M = N$. We suspect that using similar construction one can show that one cannot derive $\mathcal{F}$-definability from $\mathcal{F}$-minimality (at least when I is strong in $\mathfrak{M}$) when $I \neq M = N$ or when $I \neq M \neq N$ but we have not verified the details.

3.3. Ultrapowers

- Throughout this section, as in the previous section, $\mathfrak{M}$ is a model of $PA(\mathcal{L})$, I is cut of $\mathfrak{M}$, and $\mathcal{F} \subseteq M^I$ is an $\mathfrak{M}$-adequate family of functions. Furthermore, $\mathcal{U}$ denotes a nonprincipal ultrafilter over $B(\mathcal{F})$.

- The ultrapower
$$\mathfrak{M}^* := \prod_{\mathcal{F},\mathcal{U}} \mathfrak{M}$$
is defined as usual, except that only functions from $\mathcal{F}$ are used in the formation of the ultrapower. More specifically, the universe of $\mathfrak{M}^*$ consists of the equivalence classes $[f]$, where $f \in \mathcal{F}$ and the equivalence relation $\sim$ at work is defined via:
$$f \sim g \Longleftrightarrow \{i \in I : f(i) = g(i)\} \in \mathcal{U}.$$
The $\mathfrak{M}^*$-operations are defined as in the usual theory of ultrapowers, i.e.,
$$[f] \oplus^{\mathfrak{M}^*} [g] = [h] \Longleftrightarrow \{i \in I : f(i) \oplus^{\mathfrak{M}} g(i) = h(i)\} \in \mathcal{U},$$

and

$$[f] \otimes^{\mathfrak{M}^*} [g] = [h] \Longleftrightarrow \{i \in I : f(i) \otimes^{\mathfrak{M}} g(i) = h(i)\} \in \mathcal{U},$$

The following proposition is proved, as in the classical Łoś theorem, by an induction on the complexity of formulae (the Skolem property of $\mathcal{F}$ is invoked in the existential step of the induction, see the proof of Proposition 3.3.4 for more detail). The amenability property is not needed in the proof, but it is crucial for later results.

Proposition 3.3.1. *For all formulas* $\varphi(x_1, \cdots, x_n)$ of $\mathcal{L}$ *and every* $f_1, \cdots, f_n$ *in* $\mathcal{F}$,

$$\mathfrak{M}^* \models \varphi([f_1], \cdots, [f_n]) \Longleftrightarrow \{i \in I : \mathfrak{M} \models \varphi(f_1(i), \cdots, f_n(i))\} \in \mathcal{U}.$$

Corollary 3.3.2. *The natural embedding* $e : \mathfrak{M} \to \mathfrak{M}^*$ *defined by*

$$m \mapsto_e [c_m]$$

is an elementary embedding.

- In light of Corollary 3.3.2, we shall identify $\mathfrak{M}$ with its isomorphic image under e and write $\mathfrak{M} \prec \mathfrak{M}^*$. Consequently, from here on we identify the element $[c_m]$ with m.

If $\mathcal{U}$ happens to be an $\mathcal{F}$-definable ultrafilter, then we can iterate the process of ultrapower formation along any linear order $\mathbb{L}$ to build the iterated ultrapower $\prod\limits_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M}$ of length $\mathbb{L}$. To describe how to iterate the ultrapower formation along an arbitrary linear order, one first inductively defines, for each natural number $n \geq 1$, an ultrafilter $\mathcal{U}^n$ on the Boolean algebra $B(\mathcal{F}_n)$. Assuming that $\mathcal{U}^n$ is already constructed and is $\mathcal{F}_n$-definable, we build $\mathcal{U}^{n+1}$ via:

$$X \in \mathcal{U}^{n+1} \Longleftrightarrow \{i_1 \in I : \{(i_2, \cdots, i_{n+1}) \in I^n : (i_1, i_2, \cdots, i_{n+1}) \in X\} \in \mathcal{U}^n\} \in \mathcal{U}.$$

Standard arguments show that $\mathcal{U}^{n+1}$ is well-defined, concentrates on $[I]^{n+1}$, and is $\mathcal{F}_{n+1}$-definable (see Claim ♣ of the proof of Theorem 3.3.6). In our applications, where $\mathcal{U}$ is $\mathcal{F}$-Ramsey (and therefore also $\mathcal{F}$-definable by Proposition 3.2.2(b)), we have the following recursion-free description of $\mathcal{U}^n$, whose proof is left to the reader.

Theorem 3.3.3. *If* $\mathcal{U}$ *is* $\mathcal{F}$*-Ramsey, then*

$$\mathcal{U}^n = \{Y \in B(\mathcal{F}_n) : \exists X \in \mathcal{U} \ \ [X]^n \subseteq Y\}.$$

- For the rest of this section assume that $\mathcal{U}$ is nonprincipal $\mathcal{F}$-definable ultrafilter over $B(\mathcal{F})$.

To describe the isomorphism type of the $\mathbb{L}$-iterated ultrapower

$$\mathfrak{M}^* := \prod_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M}$$

one can either use a direct limit construction (as formulated by Kunen [Ku], and often used in set theoretic literature) or, equivalently, one can take the following route.

- Let Υ be the set of terms τ of the form $f(l_1, \cdots, l_n)$, where $n \in \mathbb{N}^+$, $f \in \mathcal{F}_n$ and $(l_1, \cdots, l_n) \in [\mathbb{L}]^n$.

- The universe M^* of $\mathfrak{M}^*$ consists of equivalence classes $\{[\tau] : \tau \in \Upsilon\}$, where the equivalence relation $\sim$ at work is defined as follows: given terms $f(a_1, \cdots, a_r)$ and $g(b_1, \cdots, b_s)$ from Υ, first let $P := \{a_1, \cdots, a_r\} \cup \{b_1, \cdots, b_s\}$ and $p := |P|$, and then relabel the elements of P in increasing order as $l_1 < \cdots < l_p$. This relabelling gives rise to increasing sequences $(j_1, j_2, \cdots, j_r)$ and $(k_1, k_2, \cdots, k_s)$ of indices between 1 and p such that

  $$a_1 = l_{j_1}, a_2 = l_{j_2}, \cdots, a_r = l_{j_r}$$

  and

  $$b_1 = l_{k_1}, b_2 = l_{k_2}, \cdots, b_s = l_{k_s}.$$

  With the relabelling at hand, we can define $\sim$ via: $f(a_1, \cdots, a_r) \sim g(b_1, \cdots, b_s)$ iff

  $$\{(i_1, \cdots, i_p) \in I^p : f(i_{j_1}, \cdots, i_{j_r}) = g(i_{k_1}, \cdots, i_{k_s})\} \in \mathcal{U}^p.$$

  We can also use the above relabelling to define the operations of $\mathfrak{M}^*$ as follows:

  $$[f(a_1, \cdots, a_r)] \oplus^{\mathfrak{M}^*} [g(b_1, \cdots, b_s)] := [v(l_1, \cdots, l_p)]$$

  and

  $$[f(a_1, \cdots, a_r)] \otimes^{\mathfrak{M}^*} [g(b_1, \cdots, b_s)] := [w(l_1, \cdots, l_p)]$$

  where v and w are maps from I^p into M defined as follows:

  $$v\,(i_1, \cdots, i_p) := f(i_{j_1}, \cdots, i_{j_r}) \oplus^{\mathfrak{M}} g(i_{k_1}, \cdots, i_{k_s});$$

$$w\,(i_1, \cdots, i_p) := f(i_{j_1}, \cdots, i_{j_r}) \otimes^{\mathfrak{M}} g(i_{k_1}, \cdots, i_{k_s}).$$

(it is easy to see that v and w members of $\mathcal{F}_p$, and that $\oplus^{\mathfrak{M}^*}$ and $\otimes^{\mathfrak{M}^*}$ are well-defined.)

The following proposition is the analogue of Proposition 3.3.1 for iterated ultrapowers. For notational reasons, Proposition 3.3.4 is stated for unary arithmetical formulas φ, but note that the general version for formulas φ with an arbitrary number of free variables follows from the version presented, since a canonical pairing function is available in arithmetic. In what follows π_1 and π_2 are the canonical projection functions satisfying $x = \langle \pi_1(x), \pi_2(x) \rangle$.

Proposition 3.3.4. *Suppose* $[f(l_1, \cdots, l_n)] \in M^*$ *and* $\varphi(x)$ *is a unary arithmetical formula. The following two conditions are equivalent:*

(a) $\mathfrak{M}^* \models \varphi([f(l_1, \cdots, l_n)])$;

(b) $\{(i_1, \cdots, i_n) \in I^n : \mathfrak{M} \models \varphi(f(i_1, \cdots, i_n)\} \in \mathcal{U}^n$.

Proof: Similar to the proof of Proposition 3.3.1, the proof is based on induction on the complexity of φ. We shall only explain the delicate part of the inductive proof which involves establishing $(b) \Rightarrow (a)$ for the existential step. In addition to the inductive hypothesis, suppose (b) holds $\varphi(x) = \exists y\ \theta(x, y)$, i.e., assume

(1) $\{(i_1, \cdots, i_n) \in I^n : \mathfrak{M} \models \exists y\ \theta(f(i_1, \cdots, i_n), y)\} \in \mathcal{U}^n$.

By the Skolem property of $\mathcal{F}_n$ (see Proposition 3.1.1(c)), there is some $g \in \mathcal{F}_n$ such that:

(2) If $\mathbf{i} := (i_1, \cdots, i_n) \in I^n$, then $\mathfrak{M} \models \exists y\ \theta(f(\mathbf{i}), y) \rightarrow \theta(f(\mathbf{i}), g(\mathbf{i}))$.

Coupling (1) and (2) yields

(3) $\{(i_1, \cdots, i_n) \in I^n : \mathfrak{M} \models \theta(f(i_1, \cdots, i_n), g(i_1, \cdots, i_n))\} \in \mathcal{U}^n$.

Let $\theta'(x) := \theta(\pi_1(x), \pi_2(x))$. Using a canonical pairing function, (3) can be reformulated as

(4) $\{(i_1, \cdots, i_n) \in I^n : \mathfrak{M} \models \theta'((h(i_1, \cdots, i_n))\} \in \mathcal{U}^n$,

where $h \in \mathcal{F}_n$ with $h(i_1, \cdots, i_n) = \langle f(i_1, \cdots, i_n), g(i_1, \cdots, i_n) \rangle$. Coupling (4) with the inductive hypothesis shows that (a) holds for $\varphi(x) = \exists y\ \theta(x, y)$.
□

- Recall that for $m \in M$, c_m is the constant m-function on I, i.e., $c_m : I \to \{m\}$. As in ultrapowers of length one, for any $l \in \mathbb{L}$, we shall identify the element $[c_m(l)]$ with m. We shall also identify $[id(l)]$ with l, where $id : I \to I$ is the identity function.

Corollary 3.3.5. *For any l_1 and l_2 in $\mathbb{L}$, $l_1 <_{\mathbb{L}} l_2$ iff $\mathfrak{M}^* \models l_1 < l_2$.*

Proof: Since $\mathcal{U}^2$ concentrates on $[I]^2$, this follows from Proposition 3.3.4 by choosing $\varphi(x)$ as $\pi_1(x) < \pi_2(x)$, and $f(i_1, i_2) = \langle i_1, i_2 \rangle$.
□

The following theorem summarizes the salient features of the iterated ultrapowers developed here. Gaifman [Ga] proved this result for models of PA^* when $I = M$, and $\mathcal{F}$ is the set of parametrically $\mathfrak{M}$-definable functions from I to M.

Theorem 3.3.6. *Suppose $\mathcal{U}$ is an $\mathcal{F}$-definable ultrafilter on $B(\mathcal{F})$, $\mathbb{L}$ is a linearly ordered set, and $\mathfrak{M}^* = \prod_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M}$. Then* (a)-(d) *below hold:*

(a) $\mathfrak{M} \prec \mathfrak{M}^*$.

(b) *For every $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$, and every $(l_1, \cdots, l_n) \in [\mathbb{L}]^n$*

$$\mathfrak{M}^* \models \varphi(l_1, l_2, \cdots, l_n) \Longleftrightarrow \{(i_1, \cdots, i_n) \in I^n : \mathfrak{M} \models \varphi(i_1, \cdots, i_n)\} \in \mathcal{U}^n.$$

(c) *$\mathbb{L}$ is a set of order indiscernibles in $\mathfrak{M}^*$.*
(d) *I is an initial segment of $\mathfrak{M}^*$, and $B(\mathcal{F}) = SSy_I(\mathfrak{M}^*)$.*

Proof: We shall verify only part (d) since the other parts follow directly from the definition of $\mathfrak{M}^*$ and Proposition 3.3.4. First, we shall verify the following claim that establishes that $\mathcal{U}^n$ is $\mathcal{F}_n$-definable for all $n \in \mathbb{N}^+$.

Claim ♣: *If $f : I \to \{0, 1\}$ with $f \in \mathcal{F}$ and $n \in \mathbb{N}^+$, then $\{i \in I : S^f_{n,i} \in \mathcal{U}^n\} \in B(\mathcal{F})$.*

The proof proceeds by induction on n. Observe that when $n = 1$ Claim ♣ is trivially true by the very definition of $\mathcal{F}$-definability. So suppose the claim is true for $n \geq 1$ and let us verify the veracity of the claim for $n + 1$ by establishing (1) below :

(1) $\{i \in I : S^f_{n+1,i} \in \mathcal{U}^{n+1}\} \in B(\mathcal{F})$.

Consider $\left(S^f_{n+1,i}\right)_{i_1} := \{(i_2, \cdots, i_n) \in I^n : (i_1, i_2, \cdots, i_n) \in S^f_{n+1,i}\}$. By the definition of $\mathcal{U}^{n+1}$, (1) is equivalent to

(2) $\{i \in I : \{i_1 \in I : \left(S^f_{n+1,i}\right)_{i_1} \in \mathcal{U}^n\} \in \mathcal{U}\} \in B(\mathcal{F})$.

Since there is a definable bijection $(a, b) \mapsto \langle a, b\rangle$ between I^2 and I, there is a function $g : I \to \{0, 1\}$ in $\mathcal{F}$ such that for all $(i, i_1) \in I^2$

$$\left(S^f_{n+1,i}\right)_{i_1} = S^g_{n,\langle i,i_1\rangle}.$$

Therefore, by the inductive assumption, we may conclude

(3) $\{i \in I : S^g_{n,i} \in \mathcal{U}^n\} \in B(\mathcal{F})$.

By (3), there is some $h \in \mathcal{F}$, such that $h : I \to \{0, 1\}$ and for all $i \in I$, $h(i) = 1$ iff $S^g_{n,i} \in \mathcal{U}^n$. Therefore (2) can be rephrased as:

(4) $\{i \in I : \{i_1 \in I : h(\langle i, i_1\rangle) = 1\} \in \mathcal{U}\} \in B(\mathcal{F})$.

But (4) is an immediate consequence of the $\mathcal{F}$-definability of $\mathcal{U}$. Since (4), (2), and (1) are all equivalent, this concludes the proof of Claim ♣

We now proceed to verify that I is a proper initial segment of $\mathfrak{M}^*$. This amounts to establishing the following claim by another induction on n.

Claim ♠: *If* $[f(l_1, \cdots, l_n)] < a$ *for some* $a \in I$, *then for some* $b < a$,

$$\mathfrak{M}^* \models [f(l_1, \cdots, l_n)] = b.$$

The case $n = 1$ of Claim ♠ is a direct consequence of the $\mathcal{F}$-completeness of $\mathcal{U}$, so we shall concentrate on the inductive case by assuming Claim ♠ holds for some $n \geq 1$. Let

$$X := \{(i_1, \cdots, i_{n+1}) \in I^{n+1} : \mathfrak{M} \models f(i_1, \cdots, i_{n+1}) < a\}.$$

Note that $X \in \mathcal{U}^{n+1}$ by the inductive assumption and Proposition 3.3.4. For each $i_1 \in I$, consider

$$(X)_{i_1} := \{(i_2, \cdots, i_{n+1}) \in I^n : (i_1, i_2, \cdots, i_n) \in X\}.$$

Using the definition of $\mathcal{U}^{n+1}$, we can rewrite $X \in \mathcal{U}^{n+1}$ as:

$$\overbrace{\{i_1 \in I : (X)_{i_1} \in \mathcal{U}^n\}}^{P} \in \mathcal{U}.$$

By Proposition 3.3.4 $[f(i_1, l_2, \cdots, l_{n+1})] < a$ for each $i_1 \in P$, so by the inductive hypothesis, there is a function $g : P \to I$ such that

$$\forall i_1 \in P\ [f(i_1, l_2, \cdots, l_{n+1})] = g(i_1) < a.$$

The key point is that the function g is a member of $\mathcal{F}$. To see this, consider the following family of elements of $B(\mathcal{F}_n)$

$$\{f^{-1}(t) \cap (X)_{i_1} : (i_1, t) \in P \times \overline{a}\}.$$

By $\mathcal{F}_n$-definability of $\mathcal{U}^n$,

$$\{(i_1, t) \in P \times \overline{a} : f^{-1}(t) \cap (X)_{i_1} \in \mathcal{U}^n\} \in B(\mathcal{F}_n).$$

This shows that $g \in \mathcal{F}$. Therefore by invoking Claim ♠ for $n = 1$, there is some $Q \subseteq P$ and some $b < a$ with $Y \in \mathcal{U}$ and $b \in I$, such that $g(x) = b$ for all $x \in Q$. This shows that

$$\overbrace{\{(i_1, \cdots, i_{n+1}) \in X : i_1 \in Q\}}^{X_Q} \in \mathcal{U}^{n+1}.$$

Since $f(i_1, \cdots, i_{n+1}) = b$ for all $(i_1, \cdots, i_{n+1}) \in X_Q$, $[f(l_1, \cdots, l_{n+1})] = b$, as desired. This concludes the proof of Claim ♠.

Finally, we establish that $B(\mathcal{F}) = SSy_I(\mathfrak{M}^*)$ for the case when $I \subsetneq M$ since the case $I = M$ corresponds to Skolem-Gaifman ultrapowers, for which the result is well-known and due to Gaifman [Ga]. To see that $B(\mathcal{F}) \subseteq SSy_I(\mathfrak{M}^*)$, suppose $S \in B(\mathcal{F})$. By the amenability property of $\mathcal{F}$, there is a function $g \in \mathcal{F}$ such that for all $i \in I$, $g(i)$ codes $S \cap \overline{i}$. This shows that for all $i \in I$, and any $l \in \mathbb{L}$,

$$i \in S \Longleftrightarrow \mathfrak{M}^* \models iE[g(l)],$$

thus $S \in SSy_I(\mathfrak{M}^*)$, as desired. Next we verify that $SSy_I(\mathfrak{M}^*) \subseteq B(\mathcal{F})$. By part (c) of Theorem 3.3.6, this amounts to showing that (i) below holds:

(i) For all $[f(l_1, \cdots, l_n)] \in M^*$, $\{i \in I : \mathfrak{M}^* \models iE[f(l_1, \cdots, l_n)] \in B(\mathcal{F})$.

We shall prove (i) by induction on n. Note that by part (c), if $n = 1$, then (i) is equivalent to

(ii) For all $[f(l_1)] \in M^*$, $\{i \in I : \overbrace{\{j \in I : \mathfrak{M} \models j\ E\ f(i)\}}^{X_i} \in \mathcal{U}\} \in B(\mathcal{F})$.

Invoking the $\mathcal{F}$-definability of $\mathcal{U}$, we obtain,

(iii) $\{i \in I : X_i \in \mathcal{U}\} \in B(\mathcal{F})$.

Therefore (i) holds for $n = 1$. The argument for $n > 1$ is identical to the above argument for $n = 1$ and uses Claim ♣.
□

Corollary 3.3.7. *Each automorphism j of $\mathbb{L}$ lifts to an automorphism $\hat{\jmath}$ of $\mathfrak{M}^*$ via*

$$\hat{\jmath}([f(l_1,\cdots,l_n)]) = [f(j(l_1),\cdots,j(l_n))].$$

Moreover, the map $j \mapsto \hat{\jmath}$ is a group embedding of $Aut(\mathbb{L})$ into $Aut(\mathfrak{M}^*)$

Proof: This result is an immediate consequence of the fact that every element of $\mathfrak{M}^*$ can be written as $[f(l_1,\cdots,l_n)]$ for an appropriate choice of $f \in \mathcal{F}$ and $l_1,\cdots,l_n$ in $\mathbb{L}$ (note that $\hat{\jmath}$ is well-defined.)
□

- *For the rest of the paper, $j \mapsto \hat{\jmath}$ will be as in Corollary* 3.3.7.

If the ultrafilter $\mathcal{U}$ of Theorem 3.3.6 (and Corollary 3.3.7) is additionally $\mathcal{F}$-Ramsey, then we can derive further information about the manner in which the behavior of $j \in Aut(\mathbb{L})$ is reflected in the behavior of $\hat{\jmath} \in Aut(\mathfrak{M}^*)$. In order to do so, let us adopt the following definitions:

- Given a linear ordering $(A,<)$ and $X \subseteq A$, a map $f : A \to A$ is *expansive on* X if $x < f(x)$ for all $x \in X$.
- $\overline{M}$ is the set of elements of M^* that are less than some element of M (note that $\overline{M} = M^*$ for Kirby-Paris ultrapowers, but $\overline{M} \neq M^*$ for Skolem-Gaifman ultrapowers). It is well-known that

$$\mathfrak{M} \preceq_c \overline{\mathfrak{M}} \preceq_e \mathfrak{M}^*.$$

Theorem 3.3.8. *Suppose $\mathcal{U}$ is $\mathcal{F}$-Ramsey and let I, $\mathbb{L}$, and $\mathfrak{M}^*$ be as in Theorem* 3.3.6.

(a) *For every $\mathcal{L}$-formula $\varphi(x_1,\cdots,x_n)$, and every $(l_1,\cdots,l_n) \in [\mathbb{L}]^n$, the following two conditions are equivalent:*
(i) $\mathfrak{M}^* \models \varphi(l_1,l_2,\cdots,l_n)$;
(ii) $\exists H \in \mathcal{U}$ *such that for all* $(a_1,\cdots,a_n) \in [H]^n$, $\mathfrak{M} \models \varphi(a_1,\cdots,a_n)$.
(b) *If $j \in Aut(\mathbb{L})$ is fixed point free, then $fix(\hat{\jmath}) = M$.*

(c) *If $j \in Aut(\mathbb{L})$ is expansive on $\mathbb{L}$, then $\hat{\jmath}$ is expansive on $M^*\backslash\overline{M}$.*

Proof:

(a) This is an immediate consequence of Theorem 3.3.3 and part (b) of Theorem 3.3.6.

(b) Every automorphism of the form $\hat{\jmath}$ fixes each $m \in M$ since $\mathcal{F}$ contains the constant map $c_m(x) = m$. To see that $\hat{\jmath}$ moves every element of $M^*\backslash M$ for a fixed point free automorphism j of $\mathbb{L}$, suppose that

(1) $[f(j(l_1), \cdots, j(l_n))] = [f(l_1, \cdots, l_n)]$,

where $[f(l_1, \cdots, l_n)] \in M^*$. Since $f \in \mathcal{F}$ and $\mathcal{U}$ is $\mathcal{F}$-canonically Ramsey, there is some $H \in \mathcal{U}$ and some $S \subseteq \{1, \cdots, n\}$ such that for all sequences $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ of elements of H,

(2) $f(x_1, \cdots, x_n) = f(y_1, \cdots, y_n) \Longleftrightarrow \forall i \in S\ (x_i = y_i)$.

Moreover, since $H^n \in \mathcal{U}^n$, by part (a) of Theorem 3.3.8,

(3) $f(l_1, \cdots, l_n) = f(k_1, \cdots, k_n) \Longleftrightarrow \forall i \in S\ (l_i = k_i)$ for all $l_1 < \cdots < l_n$ and $k_1 < \cdots < k_n$ from $\mathbb{L}$.

Coupled with (1), (3) shows that $S = \emptyset$. So f must be *constant* on H, thereby showing that $[f(l_1, \cdots, l_n)] \in M$, as desired.

(c): Suppose j is an expansive automorphism of $\mathbb{L}$. We wish to show that

(i) $[f(l_1, \cdots, l_n)] \notin \overline{M} \Longrightarrow [f(l_1, \cdots, l_n)] < [f(j(l_1), \cdots, j(l_n))]$.

Let $k_i := j(l_i)$ for $1 \leq i \leq n$. We shall first establish (i) under the assumption that $l_n < k_1$, and then we shall explain why the general case can be reduced to this special case. So suppose $[f(l_1, \cdots, l_n)] \notin \overline{M}$, and

$$l_1 < \cdots < l_n < k_1 < \cdots < k_n.$$

Assume to the contrary that the conclusion of (i) is false. Thanks to part (a), we know that $[f(l_1, \cdots, l_n)] \neq [f(k_1, \cdots, k_n)]$, therefore we are entitled to

(ii) $[f(l_1, \cdots, l_n)] > [f(k_1, \cdots, k_n)]$.

(ii), coupled with Theorem 3.3.6(a) shows that

$$\{(i_1, \cdots, i_n) \in I^{2n} : \mathfrak{M} \models \varphi(i_1, \cdots, i_{2n})\} \in \mathcal{U}^{2n},$$

where

$$\varphi(i_1, \cdots, i_{2n}) := \left(\left(\bigwedge_{s=1}^{2n-1} i_s < i_{s+1} \right) \rightarrow f(i_1, \cdots, i_n) > f(i_{n+1}, \cdots, i_{2n}) \right).$$

Coupled with the $\mathcal{F}$-Ramsey property of $\mathcal{U}$, this implies that there is some $H \in \mathcal{U}$ such that for any increasing sequence $a_1 < \cdots < a_n < a_{n+1} < \cdots < a_{2n}$ from H,

$$f(a_1, \cdots, a_n) > f(a_{n+1}, \cdots, a_{2n}).$$

For $i = 1, \cdots, n$, let $h_i :=$ the i-th member of H (in its natural ordering), and let $m := f(h_1, \cdots, h_n) \in M$. Then for all sequences $a_1 < \cdots < a_n$ from $K := H \backslash \{h_1, \cdots, h_n\}$,

$$m > f(a_1, \cdots, a_n).$$

But $K \in \mathcal{U}$, and therefore $K^n \in \mathcal{U}^n$, so by part (a) of the theorem,

$$m > [f(l_1, \cdots, l_n)].$$

This contradicts the assumption that $[f(l_1, \cdots, l_n)] \notin \overline{M}$, thereby concluding the proof of (i) for the special case $l_n < k_1$. We now explain how to handle the general case, where the sets $\{l_1, \cdots, l_n\}$ and $\{k_1, \cdots, k_n\}$ might be "entangled". Let $P := \{l_1, \cdots, l_n, k_1, \cdots, k_n\}$ and $p := |P|$. It is easy to see that

$$n + 1 \leq p \leq 2n.$$

We can relabel the elements of P in increasing order as $c_1 < \cdots < c_p$. This relabelling gives rise to two increasing sequences $(j_1, j_2, \cdots, j_n)$ and $(s_1, s_2, \cdots, s_n)$ of indices between 1 and p satisfying

$$l_1 = c_1 = c_{j_1},\ l_2 = c_{j_2}, \cdots,\ l_n = c_{j_n}$$

and

$$k_1 = c_{s_1}, k_2 = c_{s_2}, \cdots,\ k_n = c_{s_n}.$$

Using this relabelling we can express (ii) as $\varphi(c_1, \cdots, c_p)$, where

$$\varphi(x_1, \cdots, x_p) := [f(x_{j_1}, \cdots, x_{j_n})] > [f(x_{s_1}, \cdots, x_{s_n})].$$

Therefore, by part (a) of the theorem, there is some $H \in \mathcal{U}$ such that

(iii) For any increasing sequence $a_1 < \cdots < a_p$ from H,

$$f(a_{j_1}, \cdots, a_{j_n}) > f(a_{s_1}, \cdots, a_{s_n}).$$

The key observation is that starting with any sequence $\mathbf{c} \in [H]^n$, we can find a finite sequence $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_t$ of members of $[H]^n$ such that for all $1 \leq i \leq t-1$, the following holds:

(A) The ordering relation between $\mathbf{c}_i$ and $\mathbf{c}_{i+1}$ is the same as the ordering relation between the sequences $(l_1, \cdots, l_n)$ and $(k_1, \cdots, k_n)$, and

(B) The last element of $\mathbf{c}_1$ is less than the first element of $\mathbf{c}_t$.

We elaborate on (A): it says that $\mathbf{c}_i \sim \mathbf{c}_{i+1}$ holds, where the binary relation $\sim$ between two members $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ of $[H]^n$ is defined by decreeing $\mathbf{a} \sim \mathbf{b}$ to hold iff for all i and j in $\{1, 2, \cdots, n\}$,

$$a_i \leq b_j \Longleftrightarrow l_i \leq k_j.$$

Therefore, by (iii),

$$f(\mathbf{c}_1) > f(\mathbf{c}_2) > \cdots > f(\mathbf{c}_t).$$

On the other hand, by the Ramsey property of $\mathcal{U}$, H can be refined to $K \subseteq H$, with $K \in \mathcal{U}$ such that either:

$(*)$ $f(\mathbf{a}) < f(\mathbf{b})$ for any two sequences $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ in $[H]^n$, with $a_n < b_1$,

or

$(**)$ $f(\mathbf{a}) > f(\mathbf{b})$ for any two sequences $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ in $[H]^n$, with $a_n < b_1$.

The key observation above, coupled with (iii), shows that $(*)$ cannot be true. Therefore $(**)$ holds, and the general case is reduced to the case considered at the beginning of the proof of part (c).
□

4. ITERATED ULTRAPOWERS AT WORK

4.1. Prelude: Embedding Aut($\mathbb{Q}$) into Aut($\mathfrak{M}$)

The existence of a *group embedding* of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$, where $\mathfrak{M}$ is a countable recursively saturated model of PA, is a corollary of a deep theorem of Schmerl [Schm-1], which states that every countable recursively saturated model with a definable β-function f (i.e., a function f coding finite sequences) is generated via f from a set of order indiscernibles of any prescribed countable order type with no last element. In this section we provide an alternative proof of the embeddability of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ that bypasses Schmerl's theorem. The methodology of the proof is quite important here since it anticipates the more elaborate proofs of the next sections.

Theorem 4.1.1. *If $\mathfrak{M}$ is a countable recursively saturated model of PA, then there is a group embedding of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$.*

Proof: By Theorem 2.2 there is a satisfaction class S of $\mathfrak{M}$. Let $\mathcal{F} := \left(M^M\right)^{(\mathfrak{M},S)}$ (i.e., the family of function from M to M that are parametrically definable in $(\mathfrak{M}, S)$). Recall that $\mathcal{F}$ is $\mathfrak{M}$-adequate by Proposition 3.1.2. Choose an ultrafilter $\mathcal{U}$ on $B(\mathcal{F})$ that is $\mathcal{F}$-definable and form the $\mathbb{Q}$-iterated ultrapower

$$(\mathfrak{M}^*, S^*) := \prod_{\mathcal{F},\mathcal{U},\mathbb{Q}} (\mathfrak{M}, S).$$

By Corollary 3.3.7 $Aut(\mathbb{Q})$ can be embedded as a subgroup of $Aut(\mathfrak{M}^*, S)$, which itself is a subgroup of $Aut(\mathfrak{M}^*)$. The proof would be complete once we show that $\mathfrak{M} \cong \mathfrak{M}^*$. This is easy to see using Theorem 2.3 by observing:
(1) $\mathfrak{M}^*$ is countable,
(2) Theorem 3.3.6(a) shows that $Th(\mathfrak{M}) = Th(\mathfrak{M}^*)$,
(3) S^* is a satisfaction class for $\mathfrak{M}^*$ and therefore $\mathfrak{M}^*$ is recursively saturated by Theorem 2.2, and
(4) $SSy(\mathfrak{M}) = SSy(\mathfrak{M}^*)$ since $\mathfrak{M}$ is end extended by $\mathfrak{M}^*$ by Theorem 3.3.6(d).
□

Remark 4.1.2. If $\mathfrak{M} = (M, <, \cdots)$ is a countable ordered model, then $Aut(\mathfrak{M})$ can be embedded into $Aut(\mathbb{Q})$. To see this, observe that every countable linear order $(M, <)$ can be embedded in such a manner in $\mathbb{Q}$ that every automorphism of $(M, <)$ extends to an automorphism of $\mathbb{Q}$ (this can be achieved by taking advantage of the fact that $\mathbb{Q}$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}$). Since $Aut(\mathfrak{M})$ is a subgroup of $Aut(M, <)$, it follows that $Aut(\mathfrak{M})$ can be embedded into $Aut(\mathbb{Q})$. If $\mathfrak{M}$ happens to be a recursively saturated model of PA, then as shown in [KKK, Theorem 4.4], there is an embedding of $Aut(\mathfrak{M})$ into $Aut(\mathbb{Q})$ whose image is *dense* in $Aut(\mathbb{Q})$ (under its natural topology, whose basic open subgroups of stabilizers of finite sets). However, it is known that $Aut(\mathbb{Q}) \ncong Aut(\mathfrak{M})$, since for example, $Aut(\mathbb{Q})$ is a divisible group (see [Gl]), but as shown in [KB], $Aut(\mathfrak{M})$ is not. Indeed, as shown in [KKK, Theorem 4.7] $Aut(\mathfrak{M})$ is not isomorphic to the automorphism group of any countable $\aleph_0$-categorical structure with the small-index property.

4.2. Schmerl's conjecture

Schmerl has conjectured[10] that the isomorphism type of every elementary submodel of a countable arithmetically saturated model $\mathfrak{M}$ of PA can

[10] This conjecture first appeared in print in the guise of a question [Kos-2, Question 2.7], see also [KS-2, Question 9]. I am grateful to Roman Kossak for forcefully bringing this question/conjecture to my attention.

be realized as *fix*(j) for some automorphism j of $\mathfrak{M}$. Kossak provided evidence for a positive answer to this conjecture by showing that every countable arithmetically saturated model of PA has continuum many nonisomorphic fixed submodels [Kos-2, Corollary 4.4], and that every countable model of PA is isomorphic to a fixed point set of some automorphism of some countable arithmetically saturated model of PA [Kos-2, Theorem 2.8]. Theorem 4.2.1 below strongly confirms Schmerl's conjecture.

Theorem 4.2.1. *Suppose $\mathfrak{M}_0$ is an elementary submodel of a countable arithmetically saturated model $\mathfrak{M}$ of PA. There is $\mathfrak{M}_1 \prec \mathfrak{M}$ with $\mathfrak{M}_0 \cong \mathfrak{M}_1$ and an embedding $j \mapsto \hat{j}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$, such that fix$(\hat{j}) = \mathfrak{M}_1$ for every fixed point free $j \in Aut(\mathbb{Q})$.*

Proof: By Proposition 3.1.2, $\mathcal{F} := (M_0^{\mathbb{N}})^{\mathfrak{M}}$ is an $\mathfrak{M}_0$-adequate family. Choose an ultrafilter $\mathcal{U}$ on $B(\mathcal{F})$ that is $\mathcal{F}$-Ramsey. Consider the countable iterated ultrapower

$$\mathfrak{M}^* := \prod_{\mathcal{F},\mathcal{U},\mathbb{Q}} \mathfrak{M}_0.$$

Recall that by Proposition 3.2.2 (b), $\mathcal{F}$-Ramsey ultrafilters are $\mathcal{F}$-definable. We claim that $\mathfrak{M}^* \cong \mathfrak{M}$. Here is where Theorem 2.3 comes into play. Recall that by Theorem 3.3.6(d),

$$SSy(\mathfrak{M}^*) = SSy(\mathfrak{M}).$$

It remains to verify the recursive saturation of $\mathfrak{M}^*$:

Lemma 4.2.2. *$\mathfrak{M}^*$ is recursively saturated.*

Proof: Suppose $\{\varphi_n(x,y) : n \in \mathbb{N}\}$ is a recursive set of formula of $\mathcal{L}_A$ such that for some parameter $b \in M^*$, the type $\Phi := \{\varphi_n(x,b) : n \in \mathbb{N}\}$ is finitely realizable in $\mathfrak{M}^*$. Choose $f \in \mathcal{F}$ and $q_1, \cdots, q_n$ in $\mathbb{Q}$ such that $b = [f(q_1, \cdots, q_n)]$, where $n \geq 1$. Also choose $\widetilde{f} \in M$ such $\widetilde{f} \restriction \mathbb{N}^n = f$. In order to exhibit a realization of Φ, fix a satisfaction S class for $\mathfrak{M}$, and let s be a nonstandard element of M for which S is s-correct. Consider the function $\widetilde{g} : [\overline{s}]^n \to M$ defined in $(\mathfrak{M}, S)$ by

$$\widetilde{g}(a_1, \cdots, a_n) := \mu x \ \forall i \leq a_n \ \ulcorner \varphi_i(x, \widetilde{f}(a_1, \cdots, a_n) \urcorner \in S.$$

Since $(\mathfrak{M}, S) \models PA^*$, $\widetilde{g}$ is coded in $\mathfrak{M}$. Let $g := \widetilde{g} \restriction \mathbb{N}^n$. To see that $g \in \mathcal{F}_n$, we need to verify that if $(a_1, \cdots, a_n) \in [\mathbb{N}]^n$, then $g(a_1, \cdots, a_n) \in M_0$. Note that for a fixed choice of $a_1, \cdots, a_n$ in $\mathbb{N}$, the above equation for $\widetilde{g}$ needs no reference to S, i.e.,

$$\widetilde{g}(a_1, \cdots, a_n) = b \Longleftrightarrow \mathfrak{M} \models b = \mu x \bigwedge_{i \leq a_n} \varphi_i(x, \widetilde{f}(a_1, \cdots, a_n)).$$

In particular, since there is some $c \in M_0$ such that $f(a_1, \cdots, a_n) = c$, and $g(a_1, \cdots, a_n)$ is defined in $\mathfrak{M}$ from c, the fact that $\mathfrak{M}_0 \prec \mathfrak{M}$ assures us that $g(a_1, \cdots, a_n) \in M_0$. It remains to show that $[g(q_1, \cdots, q_n)]$ realizes Φ. This is easy to see, since for any fixed $n_0 \in \mathbb{N}$,

$$\{i \in \mathbb{N} : i \geq n_0\}^n \in \mathcal{U}^n.$$

□

Going back to the proof of Theorem 4.2.1, let θ be an isomorphism between $\mathfrak{M}^*$ and $\mathfrak{M}$ and let $\mathfrak{M}_1$ be the image of $\mathfrak{M}_0$ under θ. By Theorem 3.3.8(b) there is an embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M}^*)$ with the property that $fix(\hat{\jmath}) = \mathfrak{M}_0$ for every fixed point free $j \in Aut(\mathbb{Q})$. The desired embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ is therefore defined by

$$j \mapsto \theta \circ \hat{\jmath} \circ \theta^{-1}.$$

□

Remark 4.2.3.

(a) As observed by the referee, the proof of Theorem 4.2.1 shows that $\mathfrak{M}_1$ is bounded in $\mathfrak{M}$ iff $\mathfrak{M}_0$ is bounded in $\mathfrak{M}$. In particular, if $\mathfrak{M}_0$ is tall, then one can arrange $\mathfrak{M}_1$ to be bounded or cofinal in $\mathfrak{M}$. Moreover, the referee has observed that the strategy of the proof of Lemma 4.2.2 can be used to show that the ultrapower $\prod\limits_{\mathcal{F},\mathcal{U}} \mathfrak{M}_0$ is recursively saturated (N.B., $I = \mathbb{N}$ here). This observation can be used to establish Lemma 4.2.2 since recursive saturation is preserved under cofinal extensions [Ka, Exercise 14.9], and for any linear order $\mathbb{L}$,

$$\prod_{\mathcal{F},\mathcal{U}} \mathfrak{M}_0 \prec_c \prod_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M}_0.$$

(b) In light of the fact that every countable recursively saturated model of PA has continuum many nonisomorphic elementary submodels, Theorem 4.2.1 offers an alternative proof of the difficult half of part (e) of Theorem 1.1 dealing with the arithmetically saturated case.

4.3. Maximal automorphisms

Clearly every definable element of a structure $\mathfrak{M}$ is invariant under any automorphism of $\mathfrak{M}$. It is also well-known that if c is an undefinable element of a countable recursively saturated model $\mathfrak{M}$ of any theory, then there is an automorphism of $\mathfrak{M}$ that moves c. But, the existence of an automorphism that moves *all* undefinable elements of a countable recursively saturated model of PA is a tall order, and as shown by Theorem 2.5, can only be arranged for countable recursively saturated models of PA in which $\mathbb{N}$ is a strong cut. Automorphisms that move every nonalgebraic element of a given structure are dubbed "maximal" in the literature[11]. Kossak and Schmerl [KS-1] refined part (c) of Theorem 1.1 by showing that every countable arithmetically saturated model $\mathfrak{M}$ of PA has a maximal automorphism j that is expansive on the tail segment of $\mathfrak{M}$ that is above all the definable elements of $\mathfrak{M}$. The following result, in turn, generalizes the Kossak-Schmerl theorem.

Theorem 4.3.1. *Suppose $\mathfrak{M}$ is a countable arithmetically saturated model of PA and M_0 is the collection of definable elements of $\mathfrak{M}$.*

(a) *There is an embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ with the property that $fix(\hat{\jmath}) = M_0$ for every fixed point free $j \in Aut(\mathbb{Q})$;*

(b) *Moreover, $\hat{\jmath}$ is expansive on $M\backslash\overline{M_0}$ if j is expansive on $\mathbb{Q}$.*

Proof: This follows directly from the proof of Theorem 4.2.1 and parts (b) and (c) of Theorem 3.3.8 by choosing $\mathfrak{M}_0$ to be the prime model of $Th(\mathfrak{M})$ (whose universe consists of the definable elements of $\mathfrak{M}$). Note that if $\mathfrak{N} \preceq \mathfrak{M}$ and $\mathfrak{N} \cong \mathfrak{M}_0$, then $\mathfrak{N} = \mathfrak{M}_0$.
□

4.4. Automorphisms that fix an initial segment

Recall from part (d) of Theorem 1.1, that if $\mathfrak{M}$ is a countable recursively saturated model of PA, and I is a proper cut of $\mathfrak{M}$, then there is an automorphism j of $\mathfrak{M}$ whose fixed point set is precisely I, provided that

$$I \prec_{strong} \mathfrak{M}.$$

The converse of the above result is also true, as announced in [KKK]; see also [E-1, Lemma A.3]. Later Kossak [Kos-1] showed that the automorphism j can be required to be expansive on $M\backslash I$. We now establish a generalization of the aforementioned results:

[11] An element c of a structure $\mathfrak{M}$ in a language $\mathcal{L}$ is *algebraic* if there is a unary first order formula $\varphi(x)$ of $\mathcal{L}$ such that the solution set of φ in $\mathfrak{M}$ is finite and contains c. Note that if $\mathfrak{M}$ is linearly ordered, then any algebraic element is already definable and therefore maximal automorphisms in this case are precisely those that move all undefinable elements. The study of maximal automorphisms in the context of general model theory was initiated by Körner [Kor]. A recent contribution is by Duby [D].

Theorem 4.4.1. *Suppose I is a proper strong elementary cut of a countable recursively saturated model $\mathfrak{M}$ of PA.*

(a) *There is an embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ with the property that fix$(\hat{\jmath}) = I$ for every fixed point free $j \in Aut(\mathbb{Q})$.*

(b) *Moreover, if j is expansive on $\mathbb{Q}$, then $\hat{\jmath}$ is expansive on $M \backslash I$.*

Proof: Let $\mathcal{A} := SSy_I(\mathfrak{M})$, $\mathfrak{M}_0$ be the model $(\mathbf{I}, X)_{X\in\mathcal{A}}$ of PA^*, and $\mathcal{F}$ be the $\mathfrak{M}_0$-adequate family of all functions from I to I that are parametrically definable in $\mathfrak{M}_0$ (note that $B(\mathcal{F}) = \mathcal{A}$). Suppose S is a satisfaction class on $\mathfrak{M}$ and let $S_I := S \cap I$. Since $\mathbf{I} \prec \mathfrak{M}$, S_I correctly computes the satisfaction predicate for all standard formulas with parameters in I. Coupled with the strength of I in $\mathfrak{M}$, this shows that S_I is a satisfaction class for $\mathbf{I}$. Choose an ultrafilter $\mathcal{U}$ on $\mathcal{A}$ that is $\mathcal{F}$-Ramsey and form the iterated ultrapower

$$(\mathfrak{M}^*, X^*)_{X\in\mathcal{A}} := \prod_{\mathcal{F},\mathcal{U},\mathbb{Q}} (\mathbf{I}, X)_{X\in\mathcal{A}}.$$

By Theorem 3.3.8(b,c) there is an embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M}^*)$ with the property that $fix(\hat{\jmath}) = I$ for every fixed point free $j \in Aut(\mathbb{Q})$, and such that $\hat{\jmath}$ is expansive on $M\backslash I$ for every expansive $j \in Aut(\mathbb{Q})$. The proof will be complete once we verify that $\mathfrak{M}$ and $\mathfrak{M}^*$ are isomorphic over I. By Theorem 2.2, $\mathfrak{M}^*$ is recursively saturated since S_I^* is a satisfaction class for $\mathfrak{M}^*$. Moreover, $SSy_I(\mathfrak{M}) = SSy_I(\mathfrak{M}^*)$ by part (d) of Theorem 3.3.6, which in turn by Theorem 2.1 implies that I is strong cut of not only $\mathfrak{M}$, but also of $\mathfrak{M}^*$. In particular I is $\mathbb{N}$-coded from above in neither $\mathfrak{M}$ nor $\mathfrak{M}^*$. It follows from Theorem 2.4 that $\mathfrak{M}$ and $\mathfrak{M}^*$ are isomorphic over I, as desired. □

4.5. A Generalization

In this section we establish a generalization of Theorems 4.3.1 and 4.4.1. In order to do so, we need to formulate a key definition:

- Suppose I is a proper cut of $\mathfrak{M}$. A subset X of M is *I-coded* in $\mathfrak{M}$, if for some $c \in M$, $X = \{(c)_i : i \in I\}$, and for all distinct i and j in I, $(c)_i \neq (c)_j$.

Clearly I is itself I-coded . It is also not hard to see that if M_0 is the set of definable elements of a recursively saturated model $\mathfrak{M}$, then M_0 is $\mathbb{N}$-coded in $\mathfrak{M}$.

Theorem 4.5.1. *Suppose I is a strong cut of a countable recursively saturated model $\mathfrak{M}$ of PA, $\mathfrak{M}_0 \prec \mathfrak{M}$ and M_0 is I-coded in $\mathfrak{M}$. Then,*

(a) *There is an embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ with the property that $fix(\hat{\jmath}) = M_0$ for every fixed point free $j \in Aut(\mathbb{Q})$;*

(b) *Moreover, if j is expansive on $\mathbb{Q}$, then $\hat{\jmath}$ is expansive on $M\backslash\overline{M_0}$.*

Proof: First observe that $I \subseteq M_0$. To see this, consider $X := I \cap M_0$. Since by Theorem 2.1, $(\mathbf{I}, X)$ is a model of PA^*, if I fails to be a subset of M_0, then $r := \min(I\backslash M_0)$ exists. But then $(r-1) \in M_0$, which implies that $r \in M_0$ since M_0 is obviously closed under addition, contradiction. Next let $\mathcal{F} := (M_0^I)^{\mathfrak{M}}$, $\mathcal{U}$ be an $\mathcal{F}$-Ramsey ultrafilter on $B(\mathcal{F})$, and let

$$\mathfrak{M}^* := \prod_{\mathcal{F},\mathcal{U},\mathbb{Q}} \mathfrak{M}_0.$$

By Theorem 3.3.8(b,c) there is an embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M}^*)$ with the property that $fix(\hat{\jmath}) = M_0$ for every fixed point free $j \in Aut(\mathbb{Q})$, and such that $\hat{\jmath}$ is expansive on $M\backslash\overline{M_0}$ for every expansive $j \in Aut(\mathbb{Q})$. Therefore, to prove Theorem 4.5.1 it suffices to prove that $\mathfrak{M}^*$ is isomorphic with $\mathfrak{M}$ via an isomorphism that is the identity on M_0. In order to do so, we need to establish two central lemmas. Here is the first one, whose proof is an elaboration of the Kossak-Kotlarski proof of Theorem 2.4.

Lemma 4.5.2. *For any $c \in M$ there is $c^* \in M^*$ which satisfies the following two conditions:*

(i) *For all formulae $\varphi(x, y)$ of $\mathcal{L}_A$, and all $i \in I$, $\mathfrak{M} \models \varphi(c, i)$ iff $\mathfrak{M}^* \models \varphi(c^*, i)$.*

(ii) $\forall i \in I$ $(c)_i = (c^*)_i$.

Proof: We begin by observing that I is $\mathbb{N}$-coded from above in neither $\mathfrak{M}^*$ nor $\mathfrak{M}$ since I is strong in $\mathfrak{M}$, and by Theorems 2.1 and 3.3.6(d) I is also strong in $\mathfrak{M}^*$. Coupled with Theorem 2.4 this yields the existence of $c^* \in M^*$ that satisfies condition (i) of Lemma 4.5.2, but we have to work harder to obtain an element c^* for which also (ii) is true. Since by Theorem 2.2(b) $\mathfrak{M}$ carries a satisfaction class, there is some $r \in M$ that codes the 1-type of c in $\mathfrak{M}$ over I, i.e., r satisfies condition (#) below:

(#) For all formulae $\varphi(x, y)$ of $\mathcal{L}_A$ with the indicated free variables,

$$\forall i \in I \quad \mathfrak{M} \models \varphi(c, i) \leftrightarrow \ulcorner\varphi(x, i)\urcorner\ E\ r.$$

Since $SSy_I(\mathfrak{M}^*) = SSy_I(\mathfrak{M})$, we can fix some $r^* \in M^*$ such that

$$\{i \in I : \mathfrak{M} \models iEr\} = \{i \in I : \mathfrak{M}^* \models iEr^*\}.$$

It is also clear that there is some $f \in M$ such that $\forall i \in I\ f(i) = (c)_i$. Therefore we can also fix some $d \in M^*$ such that $(d)_i = (c)_i$ for all $i \in I$. The following claim lies at the heart of the proof.

Claim $(*)$ *Let $\langle \varphi_n(x, y) : n \in \mathbb{N}\rangle$ be a recursive enumeration of formulas of $\mathcal{L}_A$ with precisely two variables. If $n_0 \in \mathbb{N}$ and $i_0 \in I$, then*

$$\mathfrak{M}^* \models \exists x\ \forall n \leq n_0\ \forall i \leq i_0\ \left[(\varphi_n(x, i) \leftrightarrow \ulcorner \varphi_n(x, i) \urcorner\ E\ r^*) \wedge (x)_i = (d)_i\right].$$

To see that the claim is true, fix $n_0 \in \mathbb{N}$ and $i_0 \in I$. Since I is closed under exponentiation, there is some $r_{n_0,i_0} \in I$ such that r_{n_0,i_0} codes

$$\{\ulcorner \varphi_n(x, i)\urcorner \in I : \mathfrak{M} \models \ulcorner \varphi_n(x, i)\urcorner\ E\ r,\ n \leq n_0,\ i \leq i_0\}.$$

Therefore

(1) $\mathfrak{M} \models \exists x\ \forall n \leq n_0\ \forall i \leq i_0\ [(\varphi_n(x, i) \leftrightarrow \ulcorner \varphi_n(x, i)\urcorner\ E\ r_{n_0,i_0}) \wedge (x)_i = (c)_i]$.

On the other hand, in light of the fact that $\mathfrak{M}_0$ is a common elementary submodel of both $\mathfrak{M}$ and $\mathfrak{M}^*$, we have

(2) $(\mathfrak{M}, a)_{a \in M_0} \equiv (\mathfrak{M}^*, a)_{a \in M_0}$.

Since $\{r_{n_0,i_0}\} \cup \{(c)_i : i \in I\} \subseteq M_0$, the claim follows immediately from coupling (1) and (2).

With Claim $(*)$ at our disposal, fix some $k \in M^*$ above I and for $n_0 \in \mathbb{N}$ let $g(n_0)$ be defined in $\mathfrak{M}^*$ as:

$$\max\{i \leq k : \exists x\ \forall n \leq n_0\ \forall i \leq k\ [(\varphi_n(x, i) \leftrightarrow \ulcorner \varphi_n(x, i)\urcorner\ E\ r^*) \wedge (x)_i = (d)_i]\}.$$

By Claim $(*)$, $g(n_0) \notin I$ for every $n_0 \in \mathbb{N}$. Since I is not $\mathbb{N}$-coded from above in $\mathfrak{M}^*$ there is some k' with $I < k' \leq k$ such that

(3) $\forall n_0 \in \mathbb{N} \quad \mathfrak{M}^* \models \exists x\ \forall n \leq n_0\ \forall i \leq k'\ \left[(\varphi_n(c, i) \leftrightarrow \ulcorner \varphi_n(x, i)\urcorner\ E\ r^*) \wedge (x)_i = (d)_i\right]$.

Finally, consider the following recursive type $\Sigma(x)$ formulated in the language of arithmetic augmented with parameters r^*, d and k':

$$\Sigma(x) := \{\forall i \leq k'\ \left[(\varphi_n(x, i) \leftrightarrow \ulcorner \varphi_n(x, i)\urcorner\ E\ r^*) \wedge (x)_i = (d)_i\right] : n \in \mathbb{N}\}.$$

By (3) $\Sigma(x)$ is finitely realizable in $\mathfrak{M}^*$. It is evident that the desired c^* is any element of M^* that realizes $\Sigma(x)$. This concludes the proof of Lemma 4.5.2.

□

Lemma 4.2.2 assures us of the recursive saturation of $\mathfrak{M}^*$ when $I = \mathbb{N}$. As far as we can see, $\mathfrak{M}^*$ need not be recursively saturated when $I \neq \mathbb{N}$ for an arbitrary $\mathfrak{M}_0 \prec \mathfrak{M}$ but, as shown in Lemma 4.5.3 below, the additional information that M_0 is I-coded allows us to establish the recursive saturation of $\mathfrak{M}^*$.

Lemma 4.5.3. $\mathfrak{M}^*$ *is recursively saturated.*

Proof: The key observation is that there is an isomorphic copy $\mathfrak{A}$ of $\mathfrak{M}_0$ that is coded into $SSy_I(\mathfrak{M})$. To see this, consider the following sets $\oplus$ and $\otimes$ in $SSy_I(\mathfrak{M})$:

$$\oplus \ := \{\langle i,j,k\rangle \in I : \mathfrak{M} \models (c)_i + (c)_j = (c)_k\}\,;$$

$$\otimes \ := \{\langle i,j,k\rangle \in I : \mathfrak{M} \models (c)_i \cdot (c)_j = (c)_k\}.$$

Let $\mathfrak{A} := (I, \oplus, \otimes)$, and define $\psi : M_0 \to I$ by $\psi((c)_i) = i$. Clearly ψ defines an isomorphism between $\mathfrak{M}_0$ and $\mathfrak{A}$. Furthermore, there is a satisfaction predicate $S_{\mathfrak{A}}$ for $\mathfrak{A}$ in $\mathcal{A} := SSy_I(\mathfrak{M})$ because if S is a satisfaction class on $\mathfrak{M}$, then

$$S_{\mathfrak{A}} := \{\ulcorner\varphi(x,i)\urcorner \in I : \ulcorner\varphi(x,(c)_i)\urcorner \in S\}$$

is a member of $\mathcal{A}$ and is a satisfaction predicate for $\mathfrak{A}$ (note that by part (a) of Theorem 2.2 this implies that $\mathfrak{M}_0$ is recursively saturated if $I \neq \mathbb{N}$). Let $\mathcal{F}_I$ be the family of functions from I into I that are parametrically definable in $(\mathbf{I}, X)_{X\in\mathcal{A}}$, and consider the iterated ultrapower

$$(\mathbf{I}^*, X^*)_{X\in\mathcal{A}} := \prod_{\mathcal{F}_I,\mathcal{U},\mathbb{Q}} (\mathbf{I}, X)_{X\in\mathcal{A}}.$$

Since $S^*_{\mathfrak{A}}$ is a satisfaction class for $\mathfrak{A}^* := (I^*, \oplus^*, \otimes^*)$, and $I^* \neq \mathbb{N}$, $\mathfrak{A}^*$ is recursively saturated by Theorem 2.2(a). This immediately shows that $\mathfrak{M}^*$ is also recursively saturated once we verify the following claim:

Claim ⧫ : $\mathfrak{M}^* \cong \mathfrak{A}^*$.

Recall that the elements of M^* are of the form $[f(q_1, \cdots, q_n)]$, where $(q_1, \cdots, q_n) \in \mathbb{Q}^n$ and $f : I^n \to M_0$ with $f \in (M_0^I)^{\mathfrak{M}}$, and the elements of I^* are of the form $[g(q_1, \cdots, q_n)]$, where $(q_1, \cdots, q_n) \in \mathbb{Q}^n$ and $g : I^n \to I$ with $g \in \mathcal{F}_I$. The desired isomorphism between $\mathfrak{M}^*$ and $\mathfrak{A}^*$ is given by the map $\pi : M^* \to I^*$ defined by

$$\pi\left([f(q_1, \cdots, q_n)]\right) := \left[\psi\left(f(q_1, \cdots, q_n)\right)\right],$$

where ψ is the isomorphism between $\mathfrak{A}$ and $\mathfrak{M}_0$.
□

We are now finally ready to show that there is an isomorphism $\theta : \mathfrak{M} \to \mathfrak{M}^*$ such that

$$\forall i \in I \ (c)_i = (\theta(c))_i.$$

Recall that I is $\mathbb{N}$-coded from above in neither $\mathfrak{M}$ nor $\mathfrak{M}^*$ (as observed at the beginning of the proof of Lemma 4.5.2). In order to build θ we shall modify the *first step* in proof of Theorem 2.4 by invoking Lemma 4.5.2 to define $\theta(c) := c^*$. By conditions (i) and (ii) of Lemma 4.5.2, the rest of the inductive proof of Theorem 2.4 can then be carried out to yield an isomorphism θ between $\mathfrak{M}_0$ and $\mathfrak{M}^*$ that is the identity on I. Note that for every $i \in I$,

$$\theta\left((c)_i\right) = (\theta(c))_{\theta(i)} = (c^*)_i = (c)_i\,.$$

This shows that $\theta(a) = a$ for all $a \in M_0$ and concludes the proof Theorem 4.5.1.
□

References

[BS] J. Barwise and J. Schlipf, *On recursively saturated models of arithmetic*, in **Model Theory and Algebra: a memorial tribute to A. Robinson** (edited by D. Saracino and V. Weispfenning), Springer Lecture Notes in Mathematics, vol. 498 (1976), pp. 42-55.

[C] P. Clote, *Ultrafilters on definable sets in arithmetic*, **Logic Colloquium '84** (edited by J.B. Paris, A. J. Wilkie, and G.M. Wilmers), North-Holland, Amsterdam, 1986.

[D] G. Duby, *Automorphisms with only infinite orbits on non-algebraic elements*, **Archives for Mathematical Logic**, vol. 42 (2003), pp. 435-447.

[E-1] A. Enayat, *From bounded arithmetic to second order arithmetic via automorphisms*, **Logic in Tehran,** Proceedings of the Workshop and Conference on Logic, Algebra, and Arithmetic (edited by A. Enayat, I. Kalantari, and M. Moniri), Lecture Notes in Logic, vol. 26, ASL and A. K. Peters, 2006.

[E-2] ______, *Automorphisms of models of bounded arithmetic,* to appear in **Fundamenta Mathematicae**.

[ER] P. Erdős and R. Rado, *A combinatorial theorem*, **Journal of the London Mathematical Society**, vol. 25 (1950), pp. 249-255.

[Ga] H. Gaifman, *Models and types of Peano's arithmetic,* **Annals of Mathematical Logic***,* vol. 9 (1976), pp. 223-306.

[Gl] A. Glass, **Ordered Permutation Groups**, London Math. Soc. Lecture Notes, Cambridge University Press, Cambridge, 1981.

[J] T. Jech, **Set Theory**, Academic Press, New York, 1978.

[JE] D. Jensen and A. Ehrenfeucht, *Some problems in elementary arithmetics*, **Fundamenta Mathematicae**, vol. 92 (1976), pp. 223-244.

[Ka] R. Kaye, **Models of Arithmetic**, Oxford Logic Guides, vol. 15, Oxford University Press, 1991.

[KM] R. Kaye and D. MacPherson (editors), **Automorphisms of First-Order Structures**, Oxford University Press, Oxford, 1994.

[KKK] R. Kaye, R. Kossak, and H. Kotlarski, *Automorphisms of recursively saturated models of arithmetic*, **Annals of Pure and Applied Logic**, vol. 55 (1991), pp.67-99.

[Ki-1] L. Kirby, **Initial Segments of Models of Arithmetic**, Ph. D. thesis, University of Manchester, 1977.

[Ki-2] ______, *Ultrafilters and types on models of arithmetic*, **Annals of Pure and Applied Logic**, vol. 27 (1984). pp.215-252.

[KP] L. Kirby and J. Paris, *Initial segments of models of Peano's axioms*, in **Lecture Notes in Mathematics**, vol. 619, Springer-Verlag, 1977, pp. 211-226.

[Kor] F. Körner, *Automorphisms moving all non-algebraic points and an application to NF*, **Journal of Symbolic Logic**, vol. 63 (1998), pp. 815-830.

[Kos-1] R. Kossak, *Exercises in back-and-forth,* **Proceedings of the 9th Easter Conference in Model Theory**, (H. Wolter, ed.), Seminarbericht 112, Humboldt Universität zu Berlin, 1991.

[Kos-2] ______, *Automorphisms of recursively saturated models of Peano arithmetic: fixed point sets*, **L.J. of IGPL**, vol. 5, No 6, pp. 787-794 (1997).

[Kos-3] ______, *Satisfaction classes and automorphisms of models of PA, in* **Logic Colloquium '96** (San Sebastián), 159–169, Lecture Notes in Logic, vol. 12, Springer, Berlin, 1998.

[KB] R. Kossak and N. Bamber, *On two questions concerning the automorphism groups of countable recursively saturated models of PA*, **Archive for Mathematical Logic**, vol. 36 (1996), pp. 73-79.

[KK] R. Kossak and H. Kotlarski, *Results on automorphisms of recursively saturated models of PA*, **Fundamenta Mathematicae**, vol. 129 (1988), no. 1, pp. 9–15.

[KS-1] R. Kossak and J. Schmerl, *Arithmetically saturated models of arithmetic*, **Notre Dame Journal of Formal Logic**, vol. 36 (1995), pp. 531-546.

[KS-2] ______, **The Structure of Models of Arithmetic**, Oxford University Press, 2006.

[Kot-1] H. Kotlarski, *On elementary cuts in recursively saturated models of arithmetic*, **Fundamenta Mathematicae**, vol. 120 (1984), pp. 205-222.

[Kot-2] ______, *Automorphisms of recursively saturated models of PA: a survey*, **Notre Dame Journal of Formal Logic**, vol. 36 (1995), pp. 205-222.

[Ku] K. Kunen, *Some applications of iterated ultrapowers in set theory*, **Annals of Mathematical Logic** vol. 1 (1970), pp. 179-227.

[Ph] R.G. Phillips, *Omitting types in arithmetic and conservative extensions*, **Victoria Symposium on Nonstandard Analysis** (Univ. Victoria, Victoria, B.C., 1972), pp. 195–202. Lecture Notes in Math., vol. 369, Springer, Berlin, 1974.

[Po] K. Potthoff, *A simple tree lemma and its application to a counterexample of Phillips*. **Arch. Math. Logik Grundlagenforsch**. vol. 18 (1976/77), no. 1-2, pp. 67–71.

[MS] R. MacDowell and E. Specker, *Modelle der Arithmetik*, In **Infinitistic Methods**, Proc. Symp. Found. Math. (Warsaw, 1959), Pergamon (1961), pp. 257-263.

[Mi] J. Mileti, *The Canonical Ramsey Theorem and Computability Theory,* **Bulletin of Symbolic Logic**, vol. 11, (2005), pp. 411–427.

[R] R. Rado, *Note on canonical partitions*, **Bulletin of London Mathematical Society**, vol. 18 (1986), pp. 123-126.

[PM] J. Paris and G. Mills, *Closure properties of countable nonstandard integers*, **Fundamenta Mathematicae** vol. 103 (1979), no. 3, pp. 205–215.

[Schl] J. Schlipf, *Toward model theory through recursive saturation*, **Journal of Symbolic Logic**, vol. 43 (1978), pp. 183-206.

[Schm-1] J. Schmerl, *Recursively saturated models generated by indiscernibles*, **Notre Dame Journal of Formal Logi**c, vol. 25 (1985), pp. 99-105.

[Schm-2] _______, *Automorphism groups of models of Peano arithmetic*, **Journal of Symbolic Logic** vol. 67 (2002), pp. 1249–1264.

[Si] S. Simpson, **Subsystems of Second Order Arithmetic**, Springer, Heidelberg 1999.

[Sm-1] C. Smoryński, *Back-and-forth in a recursively saturated model of arithmetic*, in **Logic Colloquium '80** (edited by D. van Dalen, D. Lascar, and J. Smiley), North-Holland, Amsterdam, 1982.

[Sm-2] ——————, *Lectures on models of arithmetic,* in **Logic Colloquium '82** (edited by G. Lolli and A. Marcja), North Holland, Amsterdam, 1984.