Model Theory of the Reflection Scheme

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- The reflection principle for ZF : for any set theoretical formula $\varphi(x)$ (possibly with parameters) there is a rank initial segment V_α of the universe that is φ-reflective, i.e., for any $s \in V_\alpha$, $\varphi(s)$ holds in the universe iff $\varphi(s)$ holds in V_α.

- Given a language $\mathcal{L}$ with a distinguished symbol $<$ for a linear order, the reflection scheme over $\mathcal{L}$, denoted $REF(\mathcal{L})$, consists of the sentence "$<$ is a linear order without a last element" plus the universal closure of formulas of the form

$$\exists x \ \forall y_1 < x \ \cdots \ \forall y_1 < x$$

$$\varphi(y_1, \cdots, y_n, v_1, \cdots, v_r)) \leftrightarrow$$

$$\varphi^{<x}(y_1, \cdots, y_n, v_1, \cdots, v_r)).$$

- The *regularity scheme* $REG(\mathcal{L})$ consists of the sentence "$<$ is a linear order with no last element" plus the universal closure of axioms of the form

$$[\forall x \ \exists y < z \ \varphi(x, y, v_1, \cdots, v_r)] \rightarrow$$

$$[\exists y < z \ \forall v \ \exists x > v \ \varphi(x, y, v_1, \cdots, v_r))].$$

- Note that every model of $REF(\mathcal{L})$ is also a model of $REG(\mathcal{L})$ (but not vice versa).

- **Examples**

1. If κ is a regular infinite cardinal, then every expansion of a κ-like linear order satisfies the regularity scheme.

2. If κ is an *uncountable* regular cardinal and $<$ is the natural order on κ, then every expansion of $(\kappa, <)$ satisfies the reflection scheme.

3. More generally, if $(X, \lhd)$ is a κ-like linear order that continuously embeds a stationary subset of κ, then any expansion of $(X, \lhd)$ satisfies the reflection scheme.

4. All instances of $REG(\mathcal{L}_{PA})$ are provable in PA, where $\mathcal{L}_{PA}$ is the language of PA. In this context $REG(\mathcal{L}_{PA})$ plus the scheme $I\Delta_0$ of bounded induction is known to be equivalent to PA.

5. ZF plus "all sets are ordinal-definable" $(\mathbf{V} = \mathbf{OD})$ proves all instances of the reflection scheme in the language of $\{<_{\mathbf{OD}}, \in\}$, where $<_{\mathbf{OD}}$ is the canonical well-ordering of the ordinal-definable sets.

6. $ZF \backslash \{$Power Set Axiom$\}$plus "all sets are constructible" $(\mathbf{V} = \mathbf{L})$ proves all instances of the reflection scheme in the language $\mathcal{L} = \{<_{\mathbf{L}}, \in\}$, where $<_{\mathbf{L}}$ is the canonical well-ordering of the constructible universe.

7. The theory T of pure linear orders with no maximum element proves every instance of $REG(\{<\})$.

- **Theorem** (Keisler) *The following are equivalent for a complete first order theory* T *formulated in the language* $\mathcal{L}$.

(1) *Some model of* T *has an e.e.e.*

(2) T *proves* $REG(\mathcal{L})$.

(3) *Every countable model of* T *has an e.e.e.*

(4) *Every countable model of* T *has an* ω_1*-like e.e.e.*

(5) T *has a* κ*-like model for some regular cardinal* κ.

- **Remarks**

1. In part (5) of Keisler's Theorem, κ cannot in general be chosen as ω_2.

2. By the MacDowell-Specker Theorem, every model of PA has an e.e.e. In contrast, it is known that every completion of ZFC has an ω_1-like model that does not have an e.e.e.

3. There is a recursive scheme Φ in the language of set theory such that: (a) every completion of $ZFC + \Phi$ has a θ-like model for any uncountable $\theta \geq \omega_1$, and (b) it is consistent (relative to $ZFC +$ "there is an ω-Mahlo cardinal") that the only completions of ZFC that have an ω_2-like model are those that satisfy Φ.

4. Rubin refined (2) $\Rightarrow$ (3) of Keisler's Theorem by showing that for any countable linear order $\mathbb{L}$, and any countable model $\mathfrak{M}_0$ of $REG(\mathcal{L})$ with definable Skolem functions, there is an elementary extension $\mathfrak{M}_{\mathbb{L}}$ of $\mathfrak{M}_0$ such that the lattice of intermediate submodels $\{\mathfrak{M} : \mathfrak{M}_0 \preceq \mathfrak{M} \preceq \mathfrak{M}_{\mathbb{L}}\}$ (ordered under $\prec$) is isomorphic to the Dedekind completion of $\mathbb{L}$.

5. Since there are continuum many nonisomorphic countable Dedekind complete linear orders, this shows that every countable complete Skolemized extension of $REG(\mathcal{L})$ *has continuum many countable nonisomorphic models.*

- **Theorem** *Suppose T is a consistent theory formulated in the language $\mathcal{L}$ such that T proves REG($\mathcal{L}$).*

1. (Chang) *If κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa$, then T has a κ^+-like model.*

2. (Jensen) *If κ is a singular strong limit cardinal and $\Box_\kappa$ holds, then T has a κ^+-like model.*

Remarks

1. The converse of Chang's Theorem is false.

2. Chang's Theorem has been recently revisited in the work of Villegas-Silva, who has employed the existence of a coarse $(\kappa, 1)$-morass (instead of $\kappa^{<\kappa} = \kappa$) to establish the conclusion of Chang's Theorem for theories T formulated in languages of cardinality κ.

3. Shelah has isolated a square principle (denoted $\square_\kappa^{b*}$) that is *equivalent* to the two-cardinal transfer principle $(\omega_1, \omega) \to (\kappa^+, \kappa)$.

New Results

- **Theorem** (Splitting Theorem). *Suppose* $\mathfrak{M} \models REG(\mathcal{L})$ *with* $\mathfrak{M} \prec \mathfrak{N}$. *Let* $\mathfrak{M}^*$ *be the submodel of* $\mathfrak{M}$ *whose universe* M^* *is the convex hull of* M *in* $\mathfrak{N}$, *i.e.,*

  $$M^* := \{x \in N : \exists y \in M \ (x <_N y)\}.$$

  Then

  $$\mathfrak{M} \preceq_{cof} \mathfrak{M}^* \preceq_e \mathfrak{N}.$$

- Suppose $\mathfrak{M}$ is a model with definable Skolem functions. $\mathfrak{M}$ is *tall* iff for every element $c \in M$, the submodel generated by c is bounded in $\mathfrak{M}$.

- **Theorem** *The following three conditions are equivalent for a model* $\mathfrak{M}$ *of* $REG(\mathcal{L})$ *with definable Skolem functions.*

(1) $\mathfrak{M}$ *is tall.*

(2) $\mathfrak{M}$ *can be written as an e.e.e. chain with no last element.*

(3) $\mathfrak{M}$ *has a cofinal recursively saturated elementary extension.*

- **Theorem** *Every tall model of* $REG(\mathcal{L})$ *has a cofinal resplendent elementary extension.*

- Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are structures with a distinguished linear order $<$, and $\mathfrak{M}$ is a submodel of $\mathfrak{N}$. $\mathfrak{N}$ is said to be a *blunt extension* of $\mathfrak{M}$ if the supremum of M in $(N, <_N)$ exists, i.e., if $\{x \in N : \forall m \in M(m <_N x)\}$ has a first element.

- **Theorem.** *Suppose $\mathfrak{M}$ is a resplendent model of $REG(\mathcal{L})$. Then there is some $\mathfrak{M}_0 \prec_e \mathfrak{M}$ such that $\mathfrak{M}_0 \cong \mathfrak{M}$. Moreover, if $\mathfrak{M}$ is a model of $REF(\mathcal{L})$, then we can further require that $\mathfrak{M}_0 \prec_e^{blunt} \mathfrak{M}$.*

- **Corollary** *Every tall model of $REF(\mathcal{L})$ has a blunt elementary extension. In particular, every model of $REF(\mathcal{L})$ of uncountable cofinality has a blunt elementary extension.*

- **Theorem A**. *The following are equivalent for a complete first order theory T formulated in the language $\mathcal{L}$ with a distinguished linear order.*

(1) *Some model of T has a blunt e.e.e.*

(2) $T \vdash REF(\mathcal{L})$.

(3) *Every countable recursively saturated countable model of T has a blunt recursively saturated e.e.e.*

(4) *T has an ω_1-like e.e.e. that continuously embeds ω_1.*

(5) *T has a κ-like model for some regular uncountable cardinal κ that continuously embeds a stationary subset of κ.*

(6) *T has a κ-like model for some regular uncountable cardinal κ that has a blunt elementary extension.*

- **Remarks**

1. In contrast with part (3) of Keisler's Theorem , it is not true in general that a countable model of the reflection scheme has a blunt e.e.e. For example, no e.e.e. of the Shepherdson-Cohen minimal model of set theory can be blunt.

2. A number of central results about stationary logic $L(aa)$ can be derived, via the 'reduction method', as corollaries of Theorem A. In particular, the countable compactness of $L(aa)$, as well as the recursive enumerability of the set of valid sentences of $L(aa)$ can be directly derived from Theorem A.

Analogue of Chang's Two-cardinal Theorem

- **Theorem B.** *Suppose* T *is a consistent theory containing* $REF(\mathcal{L})$, *and* κ *is a regular cardinal with* $\kappa = \kappa^{<\kappa}$. *Then* T *has a* κ^+*-like model that continuously embeds the stationary subset* $\{\alpha < \kappa^+ : cf(\alpha) = \kappa\}$ *of* κ^+.

Open Questions

Question 1. *In the presence of the continuum hypothesis, is it true that* ω_1 *can be replaced by* ω_2 *in part (4) of Theorem A?*

Question 2. *Let* $\kappa \rightarrow_{c.u.b.} \theta$ *abbreviate the transfer relation "every sentence with a* κ*-like model that continuously embeds a stationary subset of* κ *also has a* θ*-like model that continuously embeds a c.u.b. subset of* θ*". Is there a model of* ZFC *in which the only inaccessible cardinals* κ *such that the transfer relation* $\kappa \rightarrow_{c.u.b.} \omega_2$ *holds are those cardinals* κ *that are* n*-subtle for each* $n \in \omega$*?*

- Notice that Theorem A implies that for all regular uncountable cardinal κ, $\kappa \rightarrow_{c.u.b.} \omega_1$. To motivate this question, first let $\kappa \rightarrow \theta$ abbreviate "every sentence with a κ-like model also has a θ-like model". The following three results suggest that Question 2 might have a positive answer:

  (1) Schmerl and Shelah showed that $\kappa \rightarrow \theta$ holds for $\theta \geq \omega_1$, if κ is n-Mahlo for each $n \in \omega$;

  (2) Schmerl proved that (relative to the consistency of an ω-Mahlo cardinal) there is a model of ZFC in which the only inaccessible cardinals κ such that $\kappa \rightarrow \omega_2$ holds are precisely those inaccessible cardinals κ that are n-Mahlo for each $n \in \omega$; and

  (3) Schmerl established that $\kappa \rightarrow_{c.u.b.} \theta$ holds for all $\theta \geq \omega_1$ if κ is n-subtle for each $n \in \omega$.

Question 3. *Can Theorem B be strengthened by (1) weakening the hypothesis $\kappa = \kappa^{<\kappa}$ to Shelah's square principle $\square_{\kappa}^{b^*}$ (mentioned in Remark 1.6.1), or (2) by using coarse $(\kappa, 1)$ morasses so as to allow T to have cardinality κ?*

Question 4. *Let $<$ be the natural order on ω^ω and suppose $(A, \lhd)$ and $(\omega^\omega, <)$ are elementarily equivalent. Does $(A, \lhd)$ have a blunt e.e.e.?*

- By a classical theorem of Ehrenfeucht,

  $$(\omega^\omega, <) \prec (\mathbf{Ord}, <).$$

  The answer to Question 4 is unknown even when A is countable.