AUTOMORPHISMS OF MODELS OF ARITHMETIC

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Skolem-Gaifman Ultrapowers (1)

- If $\mathfrak{M}$ has definable Skolem functions, then we can form the *Skolem ultrapower*

$$\mathfrak{M}^* = \prod_{\mathcal{F},\mathcal{U}} \mathfrak{M}$$

as follows:

(a) Let $\mathcal{B}$ be the Boolean algebra of $\mathfrak{M}$-definable subsets of M, and $\mathcal{U}$ be an ultrafilter over $\mathcal{B}$.

(b) Let $\mathcal{F}$ be the family of functions from M into M that are parametrically definable in $\mathfrak{M}$.

(c) The universe of $\mathfrak{M}^*$ is

$$\{[f] : f \in \mathcal{F}\},$$

where

$$f \sim g \Longleftrightarrow \{m \in M : f(m) = g(m)\} \in \mathcal{U}$$

Skolem-Gaifman Ultrapowers (2)

- **Theorem** (MacDowell-Specker) *Every model of PA has an elementary end extension.*

- **Proof:** Construct $\mathcal{U}$ with the property that every definable map with bounded range is constant on a member of $\mathcal{U}$ (this is similar to building a p-point in $\beta\omega$ using CH). Then,

$$\mathfrak{M} \prec_e \prod_{\mathcal{F},\mathcal{U}} \mathfrak{M}..$$

- For each parametrically definable $X \subseteq M$, and $m \in M$,

$$(X)_m = \{x \in M : \langle m, x \rangle \in X\}.$$

- $\mathcal{U}$ is an *iterable* ultrafilter if for every $X \in \mathcal{B}$, $\{m \in M : (X)_m \in \mathcal{U}\}$ is definable.

Skolem-Gaifman Ultrapowers (3)

- **Theorem** (Gaifman)

  (1) *If $\mathcal{U}$ is iterable, and $\mathbb{L}$ is a linear order, then*

  $$\mathfrak{M} \prec_{e,cons} \prod_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M} = \mathfrak{M}^*_{\mathbb{L}}.$$

  (2) *Moreover, if $\mathcal{U}$ is a 'Ramsey ultrafilter' over $\mathfrak{M}$, then there is isomorphism*

  $$j \longmapsto \widehat{\jmath}$$

  between Aut($\mathbb{L}$) and Aut($\mathfrak{M}^*_{\mathbb{L}}; M$) such that

  $$\mathit{fix}(\widehat{\jmath}) = M$$

  for every fixed-point-free j.

Schmerl's Generalization

- **Theorem** *The following are equivalent for a group G.*

  (a) $G \leq Aut(\mathbb{L})$ *for some linear order* $\mathbb{L}$.

  (b) G *is left-orderable.*

  (c) $G \cong Aut(\mathfrak{A})$ *for some linearly ordered structure* $\mathfrak{A} = (A, <, \cdots)$.

  (d) $G \cong Aut(\mathfrak{M})$ *for some* $\mathfrak{M} \models PA$.

  (e) $G \cong Aut(\mathbb{F})$ *for some ordered field* $\mathbb{F}$.

- Schmerl's methodology: Using a combinatorial theorem of Abramson-Harrington/Nešteřil-Rödl to refine Gaifman's techniques.

Countable Recursively Saturated Models (1)

- **Theorem** (Schlipf). *Every countable recursively saturated model has continuum many automorphisms.*

- **Theorem**. (Smoryński) *If* $\mathfrak{M}$ *is a countable recursively saturated model of* PA *and* I *is a cut of* $\mathfrak{M}$ *that is closed under exponentiation, then for some* $j \in Aut(\mathfrak{M})$, I *is the longest initial segment of* $\mathfrak{M}$ *that is pointwise fixed by* j.

- **Key Lemma** (also discovered by Kotlarski and Vencovská): *Suppose* $a, b, c \in M$ *are such that* $\forall x < 2^{2^c}$, $(\mathfrak{M}, x, a) \equiv (\mathfrak{M}, x, b)$.

  Then $\forall a' \in M \ \exists b' \in M$ *such that* $\forall x < c$, $(\mathfrak{M}, x, a, a') \equiv (\mathfrak{M}, x, b, b')$.

Countable Recursively Saturated Models (2)

- **Theorem** (Schmerl)

  (1) *If a countable recursively saturated model $\mathfrak{M}$ is equipped with a 'β-function'' β, then for any countable linear order $\mathbb{L}$ without a last element, $\mathfrak{M}$ is generated by a set of indiscernibles of order-type $\mathbb{L}$ (via β).*

  (2) *Consequently, there is a group embedding from $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$.*

- **Question.** Can Smoryński's theorem be combined with part (2) of Schmerl's theorem?

Paris-Mills Ultrapowers

- The *index set* is of the form

$$\overline{c} = \{0, 1, \cdots, c-1\}$$

for some nonstandard m in $\mathfrak{M}$.

- The family of functions used, denoted $\mathcal{F}$ is $({}^{\overline{c}}M)^{\mathfrak{M}}$.

- The Boolean algebra at work will be denoted $\mathcal{P}^{\mathfrak{M}}(\overline{c})$.

- This type of ultrapower was first considered by Paris and Mills to show that one can arrange a model of PA in which there is an externally countable nonstandard integer H such that the external cardinality of $Superexp(2, H)$ is of any prescribed infinite cardinality.

More on Ultrafilters

- A filter $\mathcal{U} \subseteq \mathcal{P}^{\mathfrak{M}}(\overline{c})$ is *canonically Ramsey* if for every $f \in \mathcal{F}_c$, and every $n \in \mathbb{N}^+$, if $f : [\overline{c}]^n \to M$, then there is some $H \in \mathcal{U}$ such that H is f-canonical;

- $\mathcal{U}$ is *I-tight* if for every $f \in \mathcal{F}_c$, and every $n \in \mathbb{N}^+$, if $f : [\overline{c}]^n \to M$, then there is some $H \in \mathcal{U}$ such either f is constant on H, or there is some $m_0 \in M \backslash I$ such that $f(\mathbf{x}) > m_0$ for all $\mathbf{x} \in [H]^n$.

- $\mathcal{U}$ is *I-conservative* if for every $n \in \mathbb{N}^+$ and every $\mathfrak{M}$-coded sequence $\langle K_i : i < c \rangle$ of subsets of $[\overline{c}]^n$ there is some $X \in \mathcal{U}$ and some $d \in M$ with $I < d \leq c$ such that $\forall i < d$ X *decides* K_i, i.e., either $[X]^n \subseteq K_i$ or $[X]^n \subseteq [\overline{c}]^n \backslash K_i$.

Desirable Ultrafilters

- **Theorem.** *$\mathcal{P}^{\mathfrak{M}}(\overline{c})$ carries a nonprincipal ultrafilter $\mathcal{U}$ satisfying the following four properties :*

(a) *$\mathcal{U}$ is I-complete;*

(b) *$\mathcal{U}$ is canonically Ramsey;*

(c) *$\mathcal{U}$ is I-tight;*

(d) $\{Card^{\mathfrak{M}}(X) : X \in \mathcal{U}\}$ is *downward cofinal in* $M \backslash I$;

(e) *$\mathcal{U}$ is I-conservative.*

Fundamental Theorem

- **Theorem.** *Suppose I is a cut closed exponentiation in a countable model of PA, $\mathbb{L}$ is a linearly ordered set, and $\mathcal{U}$ satisfies the five properties of the previous theorem. One can use $\mathcal{U}$ to build a an elementary $\mathfrak{M}^*_{\mathbb{L}}$ of $\mathfrak{M}$ that satisfies the following:*

(a) $I \subseteq_e \mathfrak{M}_{\mathbb{L}}$ *and* $SSy_I(\mathfrak{M}_{\mathbb{L}}) = SSy_I(\mathfrak{M})$.

(b) $\mathbb{L}$ *is a set of indiscernibles in* $\mathfrak{M}^*_{\mathbb{L}}$;

(c) *Every* $j \in Aut(\mathbb{L})$ *induces an automorphism* $\widehat{j} \in Aut(\mathfrak{M}^*_{\mathbb{L}})$ *such that* $j \mapsto \widehat{j}$ *is a group embedding of* $Aut(\mathbb{L})$ *into* $Aut(\mathfrak{M}^*_{\mathbb{L}})$;

(d) *If* $j \in Aut(\mathbb{L})$ *is nontrivial, then* $I_{fix}(\widehat{j}) = I$*;*

(e) *If* $j \in Aut(\mathbb{L})$ *is fixed point free, then*

$$\text{fix}(\widehat{j}) = M.$$

Combining Smoryński and Schmerl

- **Theorem.** *Suppose I is a cut closed under exponentiation in a countable recursively saturated model $\mathfrak{M}$ of PA, and $\mathfrak{M}^*$ is a cofinal countable elementary extension of $\mathfrak{M}$ such that $I \subseteq_e \mathfrak{M}^*$ with $SSy_I(\mathfrak{M}) = SSy_I(\mathfrak{M}^*)$. Then $\mathfrak{M}$ and $\mathfrak{M}^*$ are isomorphic over I.*

- **Theorem.** *Suppose $\mathfrak{M}$ is a countable recursively saturated model of PA and I is a cut of $\mathfrak{M}$ that is closed under exponentiation. There is a group embedding $j \longmapsto \widehat{\jmath}$ from $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ such that for every nontrivial $j \in Aut(\mathbb{Q})$ the longest initial segment of $\mathfrak{M}$ that is pointwise fixed by $\widehat{\jmath}$ is I. Moreover, for every fixed point free $j \in Aut(\mathbb{Q})$, the fixed point set of $\widehat{\jmath}$ is isomorphic to $\mathfrak{M}$.*

A Characterization of $I\Delta_0 + Exp + B\Sigma_1$

- $B\Sigma_1$ is the Σ_1-collection scheme consisting of the universal closure of formulae of the form, where φ is a Δ_0-formula:

$$[\forall x < a\ \exists y\ \varphi(x, y)] \rightarrow [\exists z\ \forall x < a\ \exists y < z\ \varphi(x, y)].$$

- $I_{fix}(j)$ is the largest initial segment of the domain of j that is pointwise fixed by j

- **Theorem** *The following two conditions are equivalent for a countable model $\mathfrak{M}$ of the language of arithmetic:*

  (1) $\mathfrak{M} \models I\Delta_0 + B\Sigma_1 + Exp$.

  (2) $\mathfrak{M} = I_{fix}(j)$ *for some nontrivial automorphism j of an end extension $\mathfrak{M}^*$ of $\mathfrak{M}$ that satisfies $I\Delta_0$.*

Strong Cuts and Arithmetic Saturation

- I is a *strong cut* of $\mathfrak{M}$ if, for each function f whose graph is coded in $\mathfrak{M}$ and whose domain includes I, there is some s in M such that for all $m \in M$, $f(m) \notin I$ iff $s < f(m)$.

- **Theorem** (Kirby-Paris) *The following are equivalent for a cut I of* $\mathfrak{M} \models PA$:

(a) I *is strong in* $\mathfrak{M}$.

(b) $(\mathbf{I}, SSy_I(\mathfrak{M})) \models ACA_0$.

- **Proposition.** *A countable recursively saturated model of* PA *is arithmetically saturated* iff $\mathbb{N}$ *is a strong cut of* $\mathfrak{M}$.

Key Results of Kaye-Kossak-Kotlarski

- **Theorem**. *Suppose $\mathfrak{M}$ is a countable recursively saturated model of PA.*

(1) *If $\mathbb{N}$ is a strong cut of $\mathfrak{M}$, then there is some $j \in Aut(\mathfrak{M})$ such that every undefinable element of $\mathfrak{M}$ is moved by j.*

(2) *If $I \prec_{e,strong} \mathfrak{M}$, then I is the fixed point set of some $j \in Aut(\mathfrak{M})$.*

A Conjecture of Schmerl

- **Conjecture** (Schmerl). *If $\mathbb{N}$ is a strong cut of countable recursively saturated model $\mathfrak{M}$ of PA, then the isomorphism types of fixed point sets of automorphisms of $\mathfrak{M}$ coincide with the isomorphism types of elementary substructures of $\mathfrak{M}$.*

- **Theorem** (Kossak).

(1) *The number of isomorphism types of fixed point sets of $\mathfrak{M}$ is either $2^{\aleph_0}$ or 1, depending on whether $\mathbb{N}$ is a strong cut of $\mathfrak{M}$, or not.*

(2) *Every countable model of PA is isomorphic to a fixed point set of some automorphism of some countable arithmetically saturated model of PA*

A New Ultrapower (1)

- Suppose $\mathfrak{M} \preceq \mathfrak{N}$, *where* $\mathfrak{M} \models PA^*$, I *is a cut of both* $\mathfrak{M}$ *and* $\mathfrak{N}$, *and* I *is strong in* $\mathfrak{N}$ *(N.B.,* I *need not be strong in* $\mathfrak{M}$*).*

- $\mathcal{F} := \left({}^{I}M\right)^{\mathfrak{N}}$.

- **Proposition.** *There is an* $\mathcal{F}$*-Ramsey ultrafilter* $\mathcal{U}$ *on* $B(\mathcal{F})$ *if* M *is countable.*

- **Theorem.** *One can build* $\mathfrak{M}^* = \prod\limits_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M}$, *and a group embedding* $j \mapsto \widehat{\jmath}$ *of* $Aut(\mathbb{Q})$ *into* $Aut(\mathfrak{M}^*)$.

A New Ultrapower (2)

- **Theorem.**

(a) $\mathfrak{M} \prec \mathfrak{M}^*$.

(b) *I is an initial segment of $\mathfrak{M}^*$, and $B(\mathcal{F}) = SSy_I(\mathfrak{M}^*)$.*

(c) *For every $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$, and every $(l_1, \cdots, l_n) \in [\mathbb{L}]^n$, the following two conditions are equivalent:*

(i) $\mathfrak{M}^* \models \varphi(l_1, l_2, \cdots, l_n)$;

(ii) *$\exists H \in \mathcal{U}$ such that for all $(a_1, \cdots, a_n) \in [H]^n$, $\mathfrak{M} \models \varphi(a_1, \cdots, a_n)$.*

(d) *If $j \in Aut(\mathbb{Q})$ is fixed point free, then fix$(\hat{\jmath}) = M$.*

(e) *If $j \in Aut(\mathbb{Q})$ is expansive on $\mathbb{Q}$, then $\hat{\jmath}$ is expansive on $M^* \backslash \overline{M}$.*

Proof of Schmerl's Conjecture (1)

- **Theorem** *Suppose $\mathfrak{M}_0$ is an elementary submodel of a countable arithmetically saturated model $\mathfrak{M}$ of PA. There is $\mathfrak{M}_1 \prec \mathfrak{M}$ with $\mathfrak{M}_0 \cong \mathfrak{M}_1$ and an embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$, such that fix($\hat{\jmath}$) $= \mathfrak{M}_1$ for every fixed point free $j \in Aut(\mathbb{Q})$.*

Proof:

(1) Let $\mathcal{F} := ({}^{\mathbb{N}}M_0)^{\mathfrak{M}}$.

(2) Build an ultrafilter $\mathcal{U}$ on $B(\mathcal{F})$ that is $\mathcal{F}$-Ramsey.

(3) $\mathfrak{M}^* := \prod_{\mathcal{F},\mathcal{U},\mathbb{Q}} \mathfrak{M}_0$.

Proof of Schmerl's Conjecture (2)

(4) $\mathfrak{M}^*$ *is recursively saturated (*key idea: $\mathfrak{M}^*$ has a satisfaction class).

(5) Therefore $\mathfrak{M}^* \cong \mathfrak{M}$.

(6) Let θ be an isomorphism between $\mathfrak{M}^*$ and $\mathfrak{M}$ and let $\mathfrak{M}_1$ be the image of $\mathfrak{M}_0$ under θ.

(7) The embedding $j \stackrel{\lambda}{\longmapsto} \widehat{j}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M}^*)$ has the property that *fix*$(\widehat{j}) = M_0$ for every fixed point free $j \in Aut(\mathbb{Q})$.

(8) The desired embedding $j \overset{\alpha}{\longmapsto} \widetilde{j}$ by:

$$\alpha = \theta^{-1} \circ \lambda \circ \theta.$$

This is illustrated by the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{M} & \overset{\widetilde{j}=\alpha(j)}{\longrightarrow} & \mathfrak{M} \\ \downarrow\theta & & \uparrow\theta^{-1} \\ \mathfrak{M}^* & \overset{\widehat{j}=\lambda(j)}{\longrightarrow} & \mathfrak{M}^* \end{array}$$