AUTOMORPHISMS AND STRONG FOUNDATIONAL SYSTEMS

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WARM-UP

- Automorphisms of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

- **Theorem** (Ehrenfeucht and Mostowski). *Given any infinite model $\mathfrak{M}$ and any linear order $\mathbb{L}$, there is an elementary extension $\mathfrak{M}_{\mathbb{L}}$ of $\mathfrak{M}$ such that*

$$Aut(\mathbb{L}) \hookrightarrow Aut(\mathfrak{M}_{\mathbb{L}}).$$

- (standard proof) Two incantations:

abracadabra (Ramsey's Theorem)

ajji majji latarrajji (Compactness Theorem).

EM WITH ONE ABRACADABRA

- $\mathfrak{M} = (M, \cdots)$ is a infinite structure, and $\mathbb{L}$ is a linear order.

- Fix a nonprincipal ultrafilter $\mathcal{U}$ over $\mathcal{P}(\mathbb{N})$.

- We shall build the $\mathbb{L}$-iterated ultrapower of $\mathfrak{M}$ modulo $\mathcal{U}$ with 'bare hands'

$$\mathfrak{M}^* := \prod_{\mathcal{U},\mathbb{L}} \mathfrak{M}.$$

A KEY DEFINITION

(REMINISCENT OF FUBINI)

- Define $\mathcal{U}^2$ as

$$\{X \subseteq \mathbb{N}^2 : \{a \in \mathbb{N} : \overbrace{\{b \in \mathbb{N} : (a,b) \in X\}}^{(X)_a} \in \mathcal{U}\} \in \mathcal{U}.$$

- More generally, define $\mathcal{U}^{n+1}$ as

$$\{X \subseteq \mathbb{N}^{n+1} : \{a \in \mathbb{N} : (X)_a \in \mathcal{U}^n\} \in \mathcal{U}\},$$

where

$$(X)_a := \{(b_1, \cdots, b_n) : (a, b_1, \cdots, b_n) \in X\}$$

BUILDING THE ITERATED ULTRAPOWER (1)

- Let Υ be the set of terms τ of the form
$$f(l_1, \cdots, l_n),$$
where $f : \mathbb{N}^n \to M$ and
$$(l_1, \cdots, l_n) \in [\mathbb{L}]^n.$$

- Given $f(l_1, \cdots, l_r)$ and $g(l_1^{'}, \cdots, l_s^{'})$ from Υ, let
$$P := \{l_1, \cdots, l_r\} \cup \{l_1^{'}, \cdots, l_s^{'}\}, \quad p := |P|,$$
and relabel the elements of P in increasing order as $\bar{l}_1 < \cdots < \bar{l}_p$. This relabelling gives rise to increasing sequences $(j_1, j_2, \cdots, j_r)$ and $(k_1, k_2, \cdots, k_s)$ from $\{1, \cdots, p\}$ such that
$$l_1 = \bar{l}_{j_1},\ l_2 = \bar{l}_{j_2},\ \cdots\ ,\ l_r = \bar{l}_{j_r}$$
and
$$l_1' = \bar{l}_{k_1},\ l_2^{'} = \bar{l}_{k_2},\ \cdots\ ,\ l_s^{'} = \bar{l}_{k_s}.$$

BUILDING THE ITERATED ULTRAPOWER (2)

- With the relabelling at hand, define:

$$f(l_1, \cdots, l_r) \sim g(l'_1, \cdots, l'_s)$$

iff

$$\{(i_1, \cdots, i_p) \in \mathbb{N}^p : f(i_{j_1}, \cdots, i_{j_r}) = g(i_{k_1}, \cdots, i_{k_s})\} \in \mathcal{U}^p$$

- The universe M^* of $\mathfrak{M}^*$ consists of equivalence classes $\{[\tau] : \tau \in \Upsilon\}$.

BUILDING THE ITERATED ULTRAPOWER (2)

- The operations and relations of $\mathfrak{M}^*$ are similarly defined, e.g.,

$$[f(l_1, \cdots, l_r)] \lhd^{\mathfrak{M}^*} [g(l_1^{'}, \cdots, l_s^{'})]$$

iff

$$\{(i_1, \cdots, i_p) \in \mathbb{N}^p : f(i_{j_1}, \cdots, i_{j_r}) \lhd^{\mathfrak{M}} g(i_{k_1}, \cdots, i_{k_s})\} \in \mathcal{U}^p.$$

PROPERTIES OF THE ITERATED ULTRAPOWER (1)

- For $m \in M$, let c_m be the constant function $c_m : \mathbb{N} \to \{m\}$. We shall identify the element $[c_m(l)]$ with m.

- We shall also identify $[id(l)]$ with l, where $id : \mathbb{N} \to \mathbb{N}$ is the identity function (WLOG $\mathbb{N} \subseteq M$).

- Therefore $M \cup \mathbb{L}$ can be viewed as a *subset* of M^*.

- **Theorem.** *For every formula* $\varphi(x_1, \cdots, x_n)$, *and every* $(l_1, \cdots, l_n) \in [\mathbb{L}]^n$, *the following are equivalent:*

(a) $\mathfrak{M}^* \vDash \varphi(l_1, l_2, \cdots, l_n)$;

(b) $\{(i_1, \cdots, i_n) \in \mathbb{N}^n : \mathfrak{M} \vDash \varphi(i_1, \cdots, i_n)\} \in \mathcal{U}^n$.

PROPERTIES OF THE ITERATED ULTRAPOWER (2)

- **Corollary 1.** *$\mathfrak{M} \prec \mathfrak{M}^*$, and $\mathbb{L}$ is a set of order indiscernibles in $\mathfrak{M}^*$.*

- **Corollary 2.** *There is a group embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{L})$ into $Aut(\mathfrak{M}^*)$ via*

$$\hat{\jmath}([f(l_1, \cdots, l_n)]) = [f(j(l_1), \cdots, j(l_n))].$$

Moreover, if j is fixed point free, then

$$fix(\hat{\jmath}) = M$$

.

SKOLEM-GAIFMAN ULTRAPOWERS

- $\mathfrak{M} \models PA$, and $\mathcal{U}$ is a nonprincipal ultrafilter on (parametrically) definable subsets of $\mathfrak{M}$.

- To allow iterations, $\mathcal{U}$ needs to be "partially codable in $\mathfrak{M}$", in the following sense:

- $\mathcal{U}$ is *iterable* if for every $\mathfrak{M}$-definable family $\langle X_m : m \in M \rangle$ of subsets of M, then the following set is definable in $\mathfrak{M}$:

$$\{m \in M : X_m \in \mathcal{U}\}.$$

- **Theorem** (Gaifman). *Let $\mathfrak{M}^*$ be the $\mathbb{Z}$-iterated ultrapower of $\mathfrak{M}$ modulo an iterable nonprincipal ultrafilter $\mathcal{U}$. Then for some $j \in Aut(\mathfrak{M}^*)$*

$$fix(j) = M.$$

REVERSING THE GAIFMAN RESULT

- I is a *strong cut* of $\mathfrak{M} \models I\Delta_0$, if for each function f whose graph is coded in M, and whose domain includes M, there is some s in M, such that for all $i \in I$,

$$f(i) \notin I \Longleftrightarrow s < f(i).$$

- **Theorem** (Kirby-Paris). *Strong cuts are models of PA.*

- **Theorem.** *If $\mathfrak{M} \models I\Delta_0$ and $j \in Aut(\mathfrak{M})$ with $fix(j) \subsetneq_e M$, then $fix(j)$ is a strong cut of $\mathfrak{M}$.*

CHARACTERIZING ACA_0

- For a cut I of $\mathfrak{M} \models I\Delta_0$, $SSy(\mathfrak{N}, I)$ is the collection of subsets of I of the form $I \cap X$, where X is a coded subset of M.

- **Theorem.** *The following two conditions are equivalent for a countable* $(\mathfrak{M}, \mathcal{A})$*:*

  (1) $(\mathfrak{M}, \mathcal{A}) \models ACA_0$.

  (2) *There is an e.e.e.* $\mathfrak{M}^*$ *of* $\mathfrak{M}$ *that possesses an automorphism* j *whose fixed point set is precisely* M*, and* $SSy(\mathfrak{M}^*, M) = \mathcal{A}$.

- (Visser Arithmetic) $VA := I\Delta_0 +$ "j is a nontrivial automorphism whose fixed point set is downward closed".

- **Theorem.** ACA_0 *is interpretable in* VA.

[ALMOST] CHARACTERIZING Z_2 (1)

- Suppose $\mathfrak{M}^* \models I\Delta_0$, and M is a cut of $\mathfrak{M}^*$. An automorphism j of $\mathfrak{M}^*$ is M-*amenable* if the fixed point set of j is precisely M, and for every formula $\varphi(x,j)$ in the language $\mathcal{L}_A \cup \{j\}$, possibly with suppressed parameters from N,

$$\{a \in M : (\mathfrak{M}^*, j) \models \varphi(a,j)\} \in SSy(\mathfrak{M}^*, M).$$

- DC is the scheme in the language of second order arithmetic consisting of formulas of the form

$$\forall n \forall X \exists Y \varphi(n,X,Y) \rightarrow \exists Z \forall n \varphi(n,(Z)_n,(Z)_{n+1}).$$

[ALMOST] CHARACTERIZING Z_2 (2)

- **Theorem**. *Suppose* $(\mathfrak{M}, \mathcal{A})$ *is a countable model of* $Z_2 + DC$. *There exists an e.e.e.* $\mathfrak{M}^*$ *of* $\mathfrak{M}$ *such that* $SSy(\mathfrak{M}^*, M) = \mathcal{A}$ *and* $\mathfrak{M}^*$ *has an* M*-amenable automorphism.*

- **Theorem**. *If* $\mathfrak{M}^* \models I\Delta_0$ *and* M *is a cut of* $\mathfrak{M}^*$ *such that* $\mathfrak{M}^*$ *has an* M*-amenable automorphism, then* $(\mathfrak{M}, SSy(\mathfrak{N}, M) \models Z_2$.

A Characterization of $I\Delta_0 + Exp + B\Sigma_1$

- $B\Sigma_1$ is the Σ_1-collection scheme consisting of the universal closure of formulae of the form, where φ is a Δ_0-formula:

$$[\forall x < a \ \exists y \ \varphi(x,y)] \rightarrow [\exists z \ \forall x < a \ \exists y < z \ \varphi(x,y)].$$

- $I_{fix}(j)$ is the largest initial segment of the domain of j that is pointwise fixed by j

- **Theorem** *The following two conditions are equivalent for a countable model $\mathfrak{M}$ of the language of arithmetic:*

  (1) $\mathfrak{M} \models I\Delta_0 + B\Sigma_1 + Exp$.

  (2) $\mathfrak{M} = I_{fix}(j)$ *for some nontrivial automorphism j of an end extension $\mathfrak{M}^*$ of $\mathfrak{M}$ that satisfies $I\Delta_0$.*

Tools for $(a) \Rightarrow (b)$:

- (1) **Theorem (**Wilkie-Paris). *Every countable model of* $I\Delta_0 + Exp + B\Sigma_1$ *has an end extension to a model of* $I\Delta_0$.

- (2) A variant of a construction of Paris-Mills: given a cut I of a countable model $\mathfrak{M} \models PA$ that is closed under exponentiation, one can fix the elements of I and 'blow-up' all elements above I to any desired cardinality in some elementary extension of $\mathfrak{M}$.

Bonus:

- A new proof, and a strengthening, of a theorem of Smoryński that characterizes cuts under exponentiation in countable recursively saturated models of PA.

ZFC+'Reflective' Mahlo Cardinals (1)

- $EST(\mathcal{L})$ [Elementary Set Theory] is obtained from the usual axiomatization of $ZFC(\mathcal{L})$ by deleting Power Set and Replacement, and adding $\Delta_0(\mathcal{L})$-Separation.

- GW_0 [Global Well-ordering] is the axiom expressing "$\lhd$ well-orders the universe".

- GW is the strengthening of GW_0 obtained by adding the following two axioms to GW_0:

  (a) $\forall x \forall y (x \in y \rightarrow x \lhd y)$;

  (b) $\forall x \exists y \forall z (z \in y \longleftrightarrow z \lhd x)$.

ZFC+'Reflective' Mahlo Cardinals (2)

- Φ is

  $\{(\kappa \text{ is } n\text{-Mahlo and } V_\kappa \prec_{\Sigma_n} \mathbf{V}) : n \in \omega\}$.

- **Theorem.** *The following are equivalent for a model* $\mathfrak{M}$ *of the language* $\mathcal{L} = \{\in, \lhd\}$.

  (a) $\mathfrak{M} = fix(j)$ *for some* $j \in Aut(\mathfrak{M}^*)$, *where* $\mathfrak{M}^* \models EST(\mathcal{L}) + GW$ *and* $\mathfrak{M}^*$ *end extends* $\mathfrak{M}^*$.

  (b) $\mathfrak{M} \models ZFC + \Phi$.

$$\boxed{\frac{I\text{-}\Delta_0}{PA} \quad \sim \quad \frac{EST(\mathcal{L})+GW}{ZFC+\Phi}}$$

A KEY EQUIVALENCE

- **Theorem.** *If* $(\mathfrak{M}, \mathcal{A}) \models GBC +$ "**Ord** *is weakly compact", then* $\mathfrak{M} \models ZFC + \Phi$.

- **Theorem.** *Every countable recursively saturated model of* $ZFC + \Phi$ *can be expanded to a model of* $GBC +$ "**Ord** *is weakly compact".*

- **Corollary.** $GBC +$ "**Ord** *is weakly compact" is a conservative extension of* $ZFC + \Phi$.

OTHER THEORIES THAT CAN BE CHARACTERIZED

- Gödel-Bernays theory of classes, augmented with a dependent choice scheme, and the sentence "**Ord** is weakly compact".

- KP^{Power} := The theory of "power admissible sets".

- The subsystem WKL_0^* of WKL_0 whose first order part is $I\Delta_0 + Exp + B\Sigma_1$.

CONNECTION WITH QUINE-JENSEN SET THEORY (1)

- The *language* of NF is $\{=, \in\}$.

- The *axioms* of NF are:

(1) Extensionality

(2) Stratified Comprehension: For each stratifiable $\varphi(x)$, "$\{x : \varphi(x)\}$ exists".

- φ is *stratifiable* if there is an integer valued function f whose domain is the set of **all** variables occurring in φ, which satisfies:

(1) $f(v) + 1 = f(w)$, whenever $(v \in w)$ is a subformula of φ;

(2) $f(v) = f(w)$, whenever $(v = w)$ is a subformula of φ.

CONNECTION WITH QUINE-JENSEN SET THEORY (2)

- Quine-Jensen set theory NFU: relax extensionality to allow urelements.

- MacLane set theory *Mac*: Zermelo set theory with Comprehension restricted to Δ_0-formulas.

- $NFU^+ := NFU$ + Infinity + Choice.

- $NFU^- := NFU$ + "V is finite" + Choice.

- **Theorem** (Jensen).

(1) Con (*Mac*) $\Rightarrow$ Con (NFU^+).

(2) Con (PA) $\Rightarrow$ Con (NFU^-).

CONNECTION WITH QUINE-JENSEN SET THEORY (3)

- $USC(X) := \{\{x\} : x \in X\}$.

- X is *Cantorian* if $card(X) = card(USC(X))$.

- X is *strongly Cantorian* if $\{\langle x, \{x\}\rangle : x \in X\}$ exists.

- $NFUA^{\pm} := NFU^{\pm}$ augmented with "every Cantorian set is strongly Cantorian".

- **Theorem.** $NFUA^{+}$ and $GBC +$ "$\mathbf{Ord}$ is weakly compact" are mutually interpretable.

- **Theorem.** $NFUA^{-}$ and ACA_0 are mutually interpretable.