Set Theory and Models of Arithmetic

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First European Set Theory Meeting

Bedlewo, July 12, 2007

PA is finite set theory!

- There is an arithmetical formula $E(x, y)$ that expresses "the x-th digit of the base 2 expansion of y is 1".

- **Theorem** (Ackermann, 1908)

- $(\mathbb{N}, E) \cong (V_\omega, \in)$.

- $\mathfrak{M} \models PA$ iff (M, E) is a model of $ZF^{-\infty}$.

Three Questions

- **Question 1.** *Is every Scott set the standard system of some model of PA*?
- **Question 2.** *Does every expansion of $\mathbb{N}$ have a conservative elementary extension*?
- **Question 3.** *Does every nonstandard model of PA have a minimal cofinal elementary extension*?
- Source: R. Kossak and J. Schmerl, **The Structure of Models of Peano Arithmetic**, Oxford University Press, 2006.

Scott Sets and Standard Systems (1)

- Suppose $\mathcal{A} \subseteq \mathcal{P}(\omega)$. $\mathcal{A}$ is a Scott set iff $(\mathbb{N}, \mathcal{A}) \models WKL_0$, equivalently:

- $\mathcal{A}$ is a Scott set iff:

  (1) $\mathcal{A}$ is a Boolean algebra;

  (2) $\mathcal{A}$ is closed under Turing reducibility;

  (3) If an infinite subset τ of $2^{<\omega}$ is coded in $\mathcal{A}$, then an infinite branch of τ is coded in $\mathcal{A}$.

- Suppose $\mathfrak{M} \models PA$.

$$SSy(\mathfrak{M}) := \{c_E \cap \omega : c \in M\}, \text{ where}$$

$$c_E := \{x \in M : \mathfrak{M} \models xEc\}.$$

Scott Sets and Standard Systems (2)

- **Theorem** (Scott 1961).

  (a) $SSy(\mathfrak{M})$ *is a Scott set.*

  (b) *All countable Scott sets can be realized as* $SSy(\mathfrak{M})$, *for some* $\mathfrak{M} \models PA$.

- **Theorem** (Knight-Nadel, 1982). *All Scott sets of cardinality at most* $\aleph_1$ *can be realized as* $SSy(\mathfrak{M})$, *for some* $\mathfrak{M} \models PA$.

- **Corollary**. *CH settles Question 1.*

McDowell-Specker-Gaifman

- $\mathfrak{M} \prec_{cons} \mathfrak{N}$, if for every parametrically definable subset X of N, $X \cap M$ is also parametrically definable.

- For models of PA, $\mathfrak{M} \prec_{cons} \mathfrak{N} \Rightarrow \mathfrak{M} \prec_{end} \mathfrak{N}$.

- **Theorem** (Gaifman, 1976). *For countable $\mathcal{L}$, every model $\mathfrak{M}$ of $PA(\mathcal{L})$ has a conservative elementary extension.*

Proof of MSG

- The desired model is a Skolem ultrapower of $\mathfrak{M}$ modulo an appropriately chosen ultrafilter.

- $\mathcal{U}$ is *complete* if every definable map with bounded range is constant on a member of $\mathcal{U}$.

- For each definable $X \subseteq M$, and $m \in M$, $(X)_m = \{x \in M : \langle m, x\rangle \in X\}$.

- $\mathcal{U}$ is an *iterable* ultrafilter if for every definable $X \in \mathcal{B}$, $\{m \in M : (X)_m \in \mathcal{U}\}$ is definable.

- There is a complete iterable ultrafilter $\mathcal{U}$ over the definable subsets of M.

Mills' Counterexample

- In 1978 Mills used a novel forcing construction to construct a countable model $\mathfrak{M}$ of $PA(\mathcal{L})$ which has no elementary end extension.

- Starting with any countable *nonstandard* model $\mathfrak{M}$ of PA and an infinite element $a \in M$, Mills' forcing produces an uncountable family $\mathcal{F}$ of functions from M into $\{m \in M : m < a\}$ such that

  (1) the expansion $(\mathfrak{M}, f)_{f \in \mathcal{F}}$ satisfies PA in the extended language employing a name for each $f \in \mathcal{F}$, and

  (2) for any distinct f and g in $\mathcal{F}$, there is some $b \in M$ such that $f(x) \neq g(x)$ for all $x \geq b$.

On Question 2

- For $\mathcal{A} \subseteq \mathcal{P}(\omega)$,

$$\Omega_{\mathcal{A}} := (\omega, +, \cdot, X)_{X \in \mathcal{A}}.$$

- **Question 2** (Blass/Mills) Does $\Omega_{\mathcal{A}}$ have a conservative elementary extension for every $\mathcal{A} \subseteq \mathcal{P}(\omega)$?

- **Reformulation:** Does $\Omega_{\mathcal{A}}$ carry an iterable ultrafilter for every $\mathcal{A} \subseteq \mathcal{P}(\omega)$?

Negative Answer to Question 2

- **Theorem A** (E, 2006) *There is* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *of power* $\aleph_1$ *such that* $\Omega_{\mathcal{A}}$ *does not carry an iterable ultrafilter.*

- Let $\mathbb{P}_{\mathcal{A}}$ denote the quotient Boolean algebra $\mathcal{A}/FIN$, where FIN is the ideal of finite subsets of ω.

- **Theorem B** (E, 2006) *There is an arithmetically closed* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *of power* $\aleph_1$ *such that forcing with* $\mathbb{P}_{\mathcal{A}}$ *collapses* $\aleph_1$.

Proof of Theorem A

- Start with a countable ω-model $(\mathbb{N}, \mathcal{A}_0)$ of second order arithmetic (Z_2) plus the choice scheme (AC) such that no nonprincipal ultrafilter on $\mathcal{A}$ is definable in $(\mathbb{N}, \mathcal{A}_0)$.

- Use $\diamondsuit_{\aleph_1}$ to elementary extend $(\mathbb{N}, \mathcal{A}_0)$ to $(\mathbb{N}, \mathcal{A})$ such that the only "piecewise coded" subsets $\mathcal{S}$ of $\mathcal{A}$ are those that are definable in $(\mathbb{N}, \mathcal{A})$.

Here $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is *piecewise coded in* $\mathcal{A}$ if for every $X \in \mathcal{A}$ there is some $Y \in \mathcal{A}$ such that

$$\{n \in \omega : (X)_n \in \mathcal{S}\} = Y,$$

where $(X)_n$ is the n-th real coded by the real X.

Proof of Theorem A, Cont'd

- The proof uses an omitting types argument, and takes advantage of a canonical correspondence between models of $Z_2 + AC$, and models of $ZFC^- +$ "all sets are finite or countable" . This yields a proof of Theorem A within $ZFC + \diamondsuit_{\aleph_1}$.

- An absoluteness theorem of Shelah can be employed to establish Theorem A within ZFC alone.

Shelah's Completeness Theorem

Theorem (Shelah, 1978). *Suppose $\mathcal{L}$ is a countable language, and $\mathfrak{t}$ is a sequence of $\mathcal{L}$-formulae that defines a ranked tree in some $\mathcal{L}$-model. Given any sentence ψ of $\mathcal{L}_{\omega_1,\omega}(Q)$, where Q is the quantifier "there exists uncountably many", there is a countable expansion $\overline{\mathcal{L}}$ of $\mathcal{L}$, and a sentence $\overline{\psi} \in \overline{\mathcal{L}}_{\omega_1,\omega}(Q)$ such that the following two conditions are equivalent:*

(1) $\overline{\psi}$ *has a model.*

(2) ψ *has a model $\mathfrak{A}$ of power $\aleph_1$ which has the property that $\mathfrak{t}^{\mathfrak{A}}$ is a ranked tree of cofinality $\aleph_1$ and every branch of $\mathfrak{t}^{\mathfrak{A}}$ is definable in $\mathfrak{A}$.*

Consequently, by Keisler's completeness theorem for $\mathcal{L}^*_{\omega_1,\omega}(Q)$, (2) is an absolute statement.

Motivation for Theorem B

- **Theorem** (Gitman, 2006). (Within $ZFC+PFA$)

  Suppose $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is arithmetically closed and $\mathbb{P}_{\mathcal{A}}$ is proper. Then $\mathcal{A}$ is the standard system of some model of PA.

- **Question** (Gitman-Hamkins).

  Is there an arithmetically closed $\mathcal{A}$ such that $\mathbb{P}_{\mathcal{A}}$ is not proper?

- Theorem B shows that the answer to the above is positive.

Open Questions (1)

Question I. *Is there* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *such that some model of* $Th(\Omega_{\mathcal{A}})$ *has no elementary end extension?*

Question II. *Suppose* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *and* $\mathcal{A}$ *is Borel.*

(a) *Does* $\Omega_{\mathcal{A}}$ *have a conservative elementary extension?*

(b) *Suppose, furthermore, that* $\mathcal{A}$ *is arithmetically closed. Is* $\mathbb{P}_{\mathcal{A}}$ *a proper poset?*

Open Questions (2)

Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{A} \subseteq \mathcal{P}(\omega)$ with $n \in \omega$, $n \geq 1$.

- $\mathcal{U}$ is $(\mathcal{A}, n)$-*Ramsey,* if for every $f : [\omega]^n \rightarrow \{0, 1\}$ whose graph is coded in $\mathcal{A}$, there is some $X \in \mathcal{U}$ such that $f \upharpoonright [X]^n$ is constant.

- $\mathcal{U}$ is $\mathcal{A}$-*Ramsey* if $\mathcal{U}$ is $(\mathcal{A}, n)$-*Ramsey* for all nonzero $n \in \omega$.

- $\mathcal{U}$ is $\mathcal{A}$-*minimal* iff for every $f : \omega \rightarrow \omega$ whose graph is coded in $\mathcal{A}$, there is some $X \in \mathcal{U}$ such that $f \upharpoonright X$ is either constant or injective.

Open Questions (3)

Theorem . *Suppose $\mathcal{U}$ is an ultrafilter on an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$.*

(a) *If $\mathcal{U}$ is $(\mathcal{A}, 2)$-Ramsey, then $\mathcal{U}$ is piecewise coded in $\mathcal{A}$.*

(b) *If $\mathcal{U}$ is both piecewise coded in $\mathcal{A}$ and $\mathcal{A}$-minimal, then $\mathcal{U}$ is $\mathcal{A}$-Ramsey.*

(c) *If $\mathcal{U}$ is $(\mathcal{A}, 2)$-Ramsey, then $\mathcal{U}$ is $\mathcal{A}$-Ramsey.*

(d) *For $\mathcal{A} = \mathcal{P}(\omega)$, the existence of an $\mathcal{A}$-minimal ultrafilter is both consistent and independent of ZFC.*

Question III. *Can it be proved in ZFC that there exists an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A}$ carries no $\mathcal{A}$-minimal ultrafilter?*