IN PRAISE OF NONSTANDARD MODELS

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OVERVIEW

• The study of nonstandard models of set theory arise in the following contexts:

  (a) Foundations of nonstandard analysis;

  (b) Generalized quantifiers;

  (c) Consistency/independence results;

  (d) Model theory of set theory.
Models of set theory are of the form $\mathcal{M} = (M, E)$, where $E = \in^\mathcal{M}$.

$\mathcal{M}$ is standard if $E$ is well-founded.

$\mathcal{M}$ is $\omega$-standard if $(\omega, <)^\mathcal{M} \cong (\omega, <)$.

**Proposition.** $\mathcal{M}$ is nonstandard iff $(\text{Ord}, \in)^\mathcal{M}$ is not well-founded.

**Proposition.** Every $\mathcal{M}$ has an elementary extension that is not $\omega$-standard.
BASICS (2)

• For $\mathcal{M} = (M, E)$, and $m \in M$,
  
  $$m_E := \{x \in M : xEm\}.$$  

• Suppose $\mathcal{M} \subseteq \mathcal{N} = (N, F)$ with $m \in M$. $\mathcal{N}$ is said to fix $m$ if $m_E = m_F$, else $\mathcal{N}$ enlarges $m$.

• $\mathcal{N}$ end extends $\mathcal{M}$ if $m_E = m_F$ for every $m \in M$.

• $\mathcal{N}$ rank extends $\mathcal{M}$ if for every $x \in N \setminus M$, and every $y \in M$, $\mathcal{N} \models \rho(x) > \rho(y)$.

• Proposition. Rank extensions are end extensions, but not vice-versa.

• Proposition. Elementary end extensions are rank extensions.
Keisler-Morely Theorem

• **Theorem** [Keisler-Morley, 1968]. Suppose $\mathcal{M}$ is a countable model of: ZFC for (a) and ZC for (b).

(a) For every prescribed linear order $\mathbb{L}$, $\mathcal{M}$ has an elementary end extension $\mathcal{N}$ which has a copy of $\mathbb{L}$ in $\text{Ord}^{\mathcal{M}}$;

(b) If $\kappa \in \text{Ord}^{\mathcal{M}}$ is a prescribed regular cardinal in the sense of $\mathcal{M}$, then there is an elementary extension $\mathcal{N} = (N, F)$ such that $\mathcal{N}$ enlarges $\kappa$ and contains a copy of $\mathbb{Q}$, but $\mathcal{N}$ fixes every member of $\kappa$.

• **Corollary.** If $\mathcal{M}$ is a countable model of $\text{Z}$, and $\kappa \in \text{Ord}^{\mathcal{M}}$ is a prescribed regular cardinal in the sense of $\mathcal{M}$, then there is an elementary extension $\mathcal{N} = (N, F)$ such that $\mathcal{N}$ enlarges $\kappa$ and is $\aleph_1$-like.
Proof of Part (b) of Keisler-Morley’s Theorem

• Let \( \mathbb{B} \) be the Boolean algebra \( \mathcal{P}(\kappa)^{\mathcal{M}} \) and let \( \mathcal{U} \) be an ultrafilter on \( \mathbb{B} \). We wish to define the (limited) ultrapower

\[ \mathcal{M}_\mathcal{U}^* \]

• Let \( \mathcal{F} \) be the family of all maps \( (\kappa^\mathcal{V})^{\mathcal{M}} \), and given \( f \) and \( g \) in \( \mathcal{F} \), define

\[ f \sim_\mathcal{U} g \iff \{ m \in M : f(m) = g(m) \} \in \mathcal{U}. \]

• The universe of \( \mathcal{M}_\mathcal{U}^* \) consists of the \( \sim_\mathcal{U} \) equivalence classes \( [f]_\mathcal{U} \) of members \( f \) of \( \mathcal{F} \). The membership relation \( F \) on \( \mathcal{M}_\mathcal{U}^* \) is defined precisely via

\[ \langle [f]_\mathcal{U}, [g]_\mathcal{U} \rangle \in F \iff \{ m \in M : \mathcal{M}_\mathcal{U}^* \models f(m) \in g(m) \} \in \mathcal{U}. \]
Proof of Part (b) of Keisler-Morley’s Theorem, Cont’d

• **Theorem** (Łoś-style theorem). For any first order formula $\varphi(x_1, \cdots, x_n)$ and any sequence $[f_1]_U, \cdots, [f_n]_U$ the following two conditions are equivalent:

1. $M^*_U \models \varphi([f_1]_U, \cdots, [f_n]_U)$;

2. $\{ m \in M : M^*_U \models \varphi(f_1(m), \cdots, f_n(m)) \} \in U$.

• **Proposition** There is a nonprincipal ultrafilter $U$ on $\mathbb{B}$ such that for $f \in F$ whose range is bounded in $\kappa$, there is some $X \in U$ such that the restriction of $f$ to $X$ is constant.
Proof of Part (b) of Keisler-Morley’s Theorem, Cont’d

• Use the Proposition to build an appropriate ultrafilter on $\mathbb{B}$, and form the ultrapower $M^*_\mathcal{U}$.

• By the Łoś-style theorem, $M^*_\mathcal{U}$ is an elementary extension of $M$, here we are identifying $[c_a]_\mathcal{U}$ with the element $a \in M$, where $c_a : \kappa \to \{a\}$.

• The fact that $\mathcal{U}$ is nonprincipal ensures that $M^*_\mathcal{U}$ is a proper extension of $M$ (since the equivalence class $[id]_\mathcal{U}$ of the identity function is not equal to any $[c_a]_\mathcal{U}$).

• Moreover, the fact that any function in $\mathcal{F}$ with bounded co-domain is constant on a member of $\mathcal{U}$, can be easily seen to imply that $M^*_\mathcal{U}$ an fixes every element of $\kappa$. 
L(Q_{\aleph_1}) via Keisler-Morely (1)

• \(L(Q)\) is the extension of first order logic obtained by adding a new (unary) quantifier \(Q\).

• **Weak models** of \(L(Q)\) are of the form \((M, q)\), where \(q \subseteq \mathcal{P}(M)\). The Tarski-style definition of satisfaction for weak-models has the new clause:

\[(M, q) \models Qx\varphi(x) \iff \{m \in M : (M, q) \models \varphi(m)\} \in q.\]

• A (strong) model of \(L(Q)\) in the \(\kappa\)-interpretation (where \(\kappa\) is an infinite cardinal) is of the form \((M, [M]^{\geq \kappa})\), where \(\kappa\) is an infinite cardinal. Here

\[[M]^{\geq \kappa} := \{X \subseteq M : |X| \geq \kappa\}.\]

• We shall write \(Q_\kappa\) when \(Q\) is interpreted in the \(\kappa\)-interpretation. \(Val(L(Q_\kappa))\) is the set of valid sentences of \(L(Q_\kappa)\).
$L(Q_{\aleph_1})$ via Keisler-Morely (2)

- **Theorem** [Mostowski 1957].

  1. $Val(L(Q_{\aleph_0}))$ is not recursively enumerable.

  2. $L(Q_{\aleph_0})$ is not countably compact.

- **Theorem** [Vaught 1964].

  1. $L(Q_{\aleph_0})$ is countably compact.

  2. $Val(L(Q_{\aleph_1}))$ is recursively enumerable.
Outline of Proof of countable compactness of $\mathcal{L}(Q_{\aleph_1})$:

Suppose $\Sigma = \{ \sigma_n : n \in \omega \}$ is a countable set of $\mathcal{L}(Q)$-sentences such that every finite subset of $\Sigma$ has a model in $\aleph_1$-interpretation.

Use compactness for first order logic to get hold of a countable non $\omega$-standard model $M$ of “enough set theory” such that there is some model $A$ in $M$ with all $n \in \omega$,

$$\forall n \in \omega \quad M \models "A \models \sigma_n".$$ 

Now use the Keisler-Morely theorem to enlarge $M$ to a model $N$ of set theory such that $(\aleph_1)^N$ is $\aleph_1$-like.

It is now routine to show that $A^N$ is a model of $\Sigma$ in the $\aleph_1$-interpretation.
A curious Independence Result

• **Theorem** [Cohen 1971]. *There is a model of ZF with an automorphism of order 2.*

• **Remarks:**

  (1) Every standard model of the extensionality axiom is rigid.

  (2) *It is known that if \( \mathcal{M} \) is a model of ZF plus (either AC, or the “the Leibniz-Myscielski axiom”), and \( j \) is an automorphism of \( \mathcal{M} \) that fixes all the ordinals of \( \mathcal{M} \), then \( j \) is the identity on \( \mathcal{M} \).*

• Consequently, Cohen’s theorem yields a new proof of the independence of the axiom of choice from ZF that necessarily uses non-standard models.
A Theorem of Friedman

- **Theorem** [Friedman, 1973]. *Every countable nonstandard model of ZF is isomorphic to a proper rank initial segment of itself.*

**Outline of proof for non \(\omega\)-standard models:**

1. Suppose \(\mathcal{M}\) is a countable non \(\omega\)-standard model of ZF, and fix a nonstandard integer \(H\) in \(\mathcal{M}\).

2. For each ordinal \(\alpha\) of \(\mathcal{M}\), let
   \[
   T_\alpha := (Th(V_\alpha, \in) \cap \{x \in \omega : x < H\})^\mathcal{M}.
   \]
   Note that \((T_\alpha \in 2^H)^\mathcal{M}\).
Proof of Friedman’s Theorem, Cont’d

(3) Invoking the replacement scheme, there is some $K \in \left(2^H\right)^M$ such that $\mathcal{M}$ satisfies \{\alpha \in \text{Ord} : T\alpha = K \text{ is cofinal in the class of ordinals}\}.

(4) By the Keisler-Morely theorem, there is an e.e.e. $\mathcal{N}$ of $\mathcal{M}$, and by (3), there is some $\beta \in \mathcal{N}\setminus\mathcal{M}$ such that $T\beta = k$.

(5) Since $\mathcal{M}$ is a non-$\omega$-standard model of $ZF$, any structure in $\mathcal{M}$ is recursively saturated.

(6) [Folklore] Any two recursively saturated countable models of set theory that are (a) elementary equivalent, and (b) have the same “standard system” are isomorphic.

(7) Therefore $(V\alpha, \in)^\mathcal{M} \simeq (V\beta, \in)^\mathcal{M}$ for some $\beta \in M$.

(8) The rest is easy!
A weak fragment of set theory

- $EST(\mathcal{L})$ [Elementary Set Theory] is obtained from the usual axiomatization of $ZFC(\mathcal{L})$ by deleting Power Set and Replacement, and adding $\Delta_0(\mathcal{L})$-Separation.

- $GW_0$ [Global Well-ordering] is the axiom expressing “$\triangleleft$ well-orders the universe”.

- $GW$ is the strengthening of $GW_0$ obtained by adding the following two axioms to $GW_0$:
  (a) $\forall x \forall y (x \in y \rightarrow x \triangleleft y)$;
  (b) $\forall x \exists y \forall z (z \in y \iff z \triangleleft x)$. 
ZFC+‘Reflective’ Mahlo Cardinals

• $\Phi$ is

\{(\kappa \text{ is } n\text{-Mahlo and } V_\kappa \prec \Sigma_n V) : n \in \omega\}.

• **Theorem** [E, 2004]. *The following are equivalent for a model $\mathcal{M}$ of the language $\mathcal{L} = \{\in, \triangleleft\}$. *

(a) $\mathcal{M} = \text{fix}(j)$ for some $j \in \text{Aut}(\mathcal{M}^*)$, where $\mathcal{M}^* \models \text{EST}(\mathcal{L}) + GW$ and $\mathcal{M}^*$ end extends $\mathcal{M}^*$.

(b) $\mathcal{M} \models ZFC + \Phi$. 
A KEY EQUIVALENCE

• **Theorem.** If \((\mathcal{M}, A) \models GBC + \text{“Ord is weakly compact”}\), then \(\mathcal{M} \models ZFC + \Phi\).

• **Theorem.** Every countable recursively saturated model of \(ZFC + \Phi\) can be expanded to a model of \(GBC + \text{“Ord is weakly compact”}\).

• **Corollary.** \(GBC + \text{“Ord is weakly compact”}\) is a conservative extension of \(ZFC + \Phi\).
Large Cardinals and Automorphisms

• Suppose \( M \) is an \( \triangleleft \)-initial segment of \( M^* := (M^*, E, \triangleleft) \). We define:

\[
SSy(M^*, M) = \{a_E \cap M : a \in M^*\},
\]

where \( a_E = \{x \in M^* : x E a\} \).

• **Theorem.** If \( j \) is an automorphism of a model \( M^* = (M^*, E, \triangleleft) \) of

\[
EST(\{\in, \triangleleft\}) + GW
\]

whose fixed point set \( M \) is a \( \triangleleft \)-initial segment of \( M^* \), and \( A := SSy(M^*, M) \), then \((M, A) \models GBC + \text{“Ord is weakly compact”}\).