Automorphisms of Models of Set Theory

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WHAT EVERY YOUNG SET THEORIST SHOULD KNOW

- **Theorem** Every well-founded model of the extensionality axiom is rigid.

- **Fraenkel-Mostowski-Specker Method of Symmetric Models:** automorphisms can be used to build various models of ZF + ATOMS in which AC fails.

- **Theorem** The action of an automorphism j of $\mathcal{M} \models \mathrm{ZFC}$ is uniquely determined by its action on $\mathbf{Ord}^{\mathcal{M}}$.

A LITTLE KNOWN GEM FROM COHEN

- **Theorem** (Cohen, 1974) There is a model of ZF that admits an automorphism of order 2.

- **Theorem** A model $\mathcal{M}$ of ZF cannot admit a nontrivial automorphism of finite order if
  (a) $\mathcal{M} \models \mathrm{AC}$.
  (b) $\mathcal{M} \models \mathrm{LM}$.

THE EHRENFEUCHT-MOSTOWSKI MACHINERY

- **Theorem** (Ehrenfeucht and Mostowski). *Given any infinite model $\mathcal{M}_0$ and any linear order $\mathbb{L}$, there is an elementary extension $\mathcal{M}_{\mathbb{L}}$ of $\mathcal{M}_0$ such that*

$$\mathrm{Aut}(\mathbb{L}) \hookrightarrow \mathrm{Aut}(\mathcal{M}_{\mathbb{L}}).$$

- **Usual Proof:** Specify an appropriate set of sentences, and build a model of them by two 'incantations':
- *abracadabra (*Ramsey's Theorem)
- *ajji majji latarrajji* (Compactness Theorem).

GAIFMAN'S PROOF OF EM THEOREM

- **Incantation:** Fix a nonprincipal ultrafilter $\mathcal{U}$.
- Use 'bare hands' to build the $\mathbb{L}$-iterated ultrapower

$$\mathcal{M}_{\mathcal{U},\mathbb{L}} := \prod_{\mathcal{U},\ \mathbb{L}} \mathcal{M}_0.$$

- $\mathcal{M}_0 \prec \mathcal{M}_{\mathcal{U},\mathbb{L}}$ and $\mathbb{L}$ is a set of order indiscernibles in $\mathcal{M}_{\mathcal{U},\mathbb{L}}$.
- There is a group embedding

$$j \mapsto \hat{\jmath}$$

of $\mathrm{Aut}(\mathbb{L})$ into $\mathrm{Aut}(\mathcal{M}_{\mathcal{U},\mathbb{L}})$ such that

$$\mathrm{fix}(\hat{\jmath}) = \mathcal{M}$$

for every fixed-point free j.

NATURAL QUESTIONS FOR T $\supseteq$ ZF

1. If T has an ω-standard model, then does T also have an ω-standard model that admits an automorphism?
2. Does T have a model that that admits an automorphism that moves all *undefinable* elements?
3. Does T have a model with an automorphism that fixes precisely a *proper rank initial segement*?
4. Does T have a model $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathbb{L})$ for any prescribed linear order $\mathbb{L}$?
5. More generally, what groups can arise as $\mathrm{Aut}(\mathcal{M})$ for $\mathcal{M} \models T$?
6. Does T have a *rigid* model?

PA IS ZF's SISTER THEORY!

- There is an arithmetical formula $E(x, y)$ that expresses "the x-th digit of the base 2 expansion of y is 1".

- **Theorem** (Ackermann 1937) $(\mathbb{N}, E) \cong (V_\omega, \in)$.

- **Theorem** (Mycielski 1964, Kaye-Wong 2006)

  $\mathrm{PA} \approx \mathrm{ZF} \backslash \{\mathrm{Infinity}\} + \neg\mathrm{Infinity} + \mathrm{TC}$.

- **Theorem** (E-Schmerl-Visser 2008)

  The above becomes false if TC is deleted.

GAIFMAN ULTRAPOWERS FOR PA

- An ultrafilter $\mathcal{U}$ on the (parametrically) definable subsets of $\mathcal{M} \models \mathrm{PA}$ is said to be "definable" if for every $\mathcal{M}$-definable family $\langle X_m : m \in M \rangle$ of subsets of M,

$$\{m \in M : X_m \in \mathcal{U}\}$$

  is $\mathcal{M}$-definable.

- Using a definable $\mathcal{U}$, and a linear order $\mathbb{L}$, one can build a "Skolem" analogue of the $\mathbb{L}$-iterated ultrapower $\mathcal{M}_{\mathbb{L}}$.

- **Theorem** (Gaifman 1967, 1976) **(a)** Every model $\mathcal{M} \models \mathrm{PA}$ carries a definable nonprincipal ultrafilter $\mathcal{U}$.

- **(b)** $\mathcal{M} \prec_{\mathrm{end}} \mathcal{M}_{\mathbb{Z}}$ and fix(j)=M for some $j \in \mathrm{Aut}(\mathcal{M}_{\mathbb{Z}})$.

BAD NEWS, GOOD NEWS

- **Theorem** (E 1983). Every model $\mathcal{M}$ of $\mathrm{ZF} + \mathrm{V} = \mathrm{HOD}$ carries a definable **Ord**-tree that has no cofinal definable branch.

- Let "Ord is WC" be the statement in class theory asserting that every "**Ord**-tree" has a cofinal branch.

- **Theorem** (E 2004) There is a recursive set of axioms Λ such that if $(\mathcal{M}, \mathcal{A}) \models \mathrm{GBC} + \mathrm{Ord}$ is WC, then $\mathcal{M} \models \mathrm{ZFC} + \Lambda$.

- **Theorem** (E 2004) Every completion of $\mathrm{ZFC} + \Lambda$ has a countable model that has an expansion to a model of $\mathrm{GBC} + \mathrm{Ord}$ is WC.

THE LEVY SCHEME Λ

- Let $\lambda_n(\kappa)$ be the sentence in set theory asserting that κ is an n-Mahlo cardinal and $V_\kappa \prec_n \mathbf{V}$.

- $\Lambda := \{\exists\kappa\ \lambda_n(\kappa) : n \in \omega\}$.

- Λ is also axiomatized by formulas of the form $\psi_{C,n} := C(x)$ is CUB $\rightarrow \exists\kappa\ C(\kappa)$ and κ is n-Mahlo.

Λ IS ROBUST

- **Theorem** If $\mathcal{M} \models \mathrm{ZFC} + \Lambda$, and $c \in M$, then $\mathbf{L}^{M}(c) \models \Lambda$.

- **Theorem** If $\mathcal{M} \models \mathrm{ZFC} + \Lambda$ and $\mathbb{P} \in M$, then $\mathcal{M}^{\mathbb{P}} \models \Lambda$.

AN ANSWER TO QUESTION 3

- **Theorem** (E 2004) Every completion T of $\mathrm{ZFC} + \Lambda$ has a model $\mathcal{M}_0$ of $T + \mathrm{ZF}(\lhd) + \mathrm{GW}$ such that $\mathcal{M}_0 \prec \mathcal{M}$ and $\mathcal{M}_0 = \mathrm{fix}(j)$ for some $j \in \mathrm{Aut}(\mathcal{M})$.

- **Theorem** (E 2004) Moreover, if j is an automorphism of $\mathcal{N} \models \mathrm{EST}$ whose fixed point set $\mathcal{M}$ is a $\lhd$-initial segment of $\mathcal{N}$, then $\mathcal{M} \vDash \mathrm{ZFC} + \Lambda$.

EST and GW

- EST(L) is obtained from the usual axiomatization of ZFC(L) by deleting Power Set and Replacement, and adding Δ_0(L)-Separation.
- GW is the conjunction of the following 3 axioms.
- (a) "$\lhd$ is a global well-ordering".
- (b) $\forall x \forall y (x \in y \to x \lhd y)$.
- (c) $\forall x \exists y \forall z (z \in y \longleftrightarrow z \lhd x)$.

- $\frac{\text{I-}\Delta_0}{\text{PA}} \quad \sim \quad \frac{\text{EST}(\in, \lhd) + \text{GW}}{\text{ZFC} + \Lambda}$

QUINE'S NF (1937)

- The *language* of NF is $\{=, \in\}$.
- The *axioms* of NF are:
- (1) Extensionality
- (2) Stratified Comprehension: For each stratifiable $\varphi(x)$, "$\{x : \varphi(x)\}$ exists".
- φ is *stratifiable* if there is an integer valued function f whose domain is the set of **all** variables occurring in φ, which satisfies:
- (1) $f(v) + 1 = f(w)$, whenever $(v \in w)$ is a subformula of φ;
- (2) $f(v) = f(w)$, whenever $(v = w)$ is a subformula of φ.

FACTS ABOUT NF

- **Theorem** (Specker 1953) AC is disprovable in NF (and therefore NF proves Infinity).

- **Theorem** (Grishin 1969)NF = NF_4, *and* Con(NF_3).

- **Theorem** (Boffa 1977) Con(NF) $\Rightarrow$ NF $\neq$ NF_3.

- **Theorem** (Boffa 1988) NF is consistent if there is a model $\mathcal{M} \models \mathrm{ZF}$ and $j \in \mathrm{Aut}(\mathcal{M})$ such that for some $m \in \mathcal{M}$

$$\mathcal{M} \models |j(m)| = |\mathcal{P}(m)|.$$

QUINE-JENSEN NFU (1969)

- Quine-Jensen set theory NFU: relax extensionality to allow urelements (atoms).
- MacLane set theory Mac is Zermelo set theory with Comprehension restricted to Δ_0-formulas.
- $\mathsf{NFU}^+ := \mathsf{NFU} +$ Infinity $+$ Choice.
- $\mathsf{NFU}^- := \mathsf{NFU} +$ "**V** is finite" $+$ Choice.
- **Theorem** (Jensen 1969) Con (Mac) $\Rightarrow$ Con (NFU^+).
- **Theorem** (Jensen 1969) Con (PA) $\Rightarrow$ Con (NFU^-).

NATURAL EXTENSIONS NFUA$^{\pm}$ of NFU

- $\mathrm{USC}(X) := \{\{x\} : x \in X\}$.

- X is *Cantorian* if $\mathrm{card}(X) = \mathrm{card}(\mathrm{USC}(X))$.

- X is *strongly Cantorian* if $\{\langle x, \{x\}\rangle : x \in X\}$ exists.

- NFUA$^{\pm}$:= NFU$^{\pm}$ augmented with "every Cantorian set is strongly Cantorian".

NFUA$^{\pm}$ AND ORTHODOX SET THEORY

- **Theorem** (Solovay, 1995) $\mathrm{Con}(\mathrm{ZFC} + \Lambda_0) \Leftrightarrow \mathrm{Con}(\mathrm{NFUA})$.

- **Theorem** (E 2004) The following are equivalent for a theory T in the language $\{\in\}$:
  (a) T is a completion of $\mathrm{ZFC} + \Lambda$.
  (b) There is a model $\mathcal{M}$ of NFUA such that T is the first order theory of ("the Cantorian part of **V**")$^{\mathcal{M}}$.

- **Theorem** (Solovay-E, 2006) The analogue of the above theorem holds for $\mathrm{ZF}_{\mathrm{fin}}$ and NFUA^{-}, in particular:

$$\mathrm{Con}(\mathrm{NFUA}^{-}) \Leftrightarrow \mathrm{Con}(\mathrm{ZF}_{\mathrm{fin}}).$$

NFUB$^{\pm}$ AND ORTHODOX SET THEORY

- **Theorem** (Holmes-Solovay 2001)
- Con$($ZFC^{-} + "there is a weakly compact cardinal"$)$ $\Leftrightarrow$ Con(NFUB^{+}).

- **Theorem** (E 2002) Con(Z$_2$) $\Leftrightarrow$ Con(NFUB^{-}).

- **Theorem** (E forthcoming) The following are equivalent for a theory T in the language of set theory.

- **(a)** T is a completion of KMC + Ord is WC+Σ^1_∞ − DC.

- **(b)** There is a model $\mathcal{M}$ of NFUB^{+} such that T is the first order theory of "canonical Kelley-Morse model of $\mathcal{M}$".

WHAT NFU KNOWS ABOUT CANTORIAN SETS

- Let $\mathrm{KP}^{\mathcal{P}}$ be the natural extension of KP in which Σ_1 is replaced by $\Sigma_1^{\mathcal{P}}$.
- For a model $\mathcal{M}$ of $\mathrm{KP}^{\mathcal{P}}$, and an automorphism j of $\mathcal{M}$, let $\mathbf{V}_{\mathrm{fix}}(\mathcal{M}, j)$ be the *longest rank initial segment* fixed by j.
- **Theorem** (E forthcoming) *The following are equivalent for a theory T in the language $\{\in\}$:*
- **(a)** T is a completion of $\mathrm{KP}^{\mathcal{P}}$.
- **(b)** T is the first order theory of $\mathbf{V}_{\mathrm{fix}}(\mathcal{M}, j)$ for some $\mathcal{M} \models \mathrm{EST}(\in, \lhd) + \mathrm{GW}$ and some $j \in \mathrm{Aut}(\mathcal{M})$ which fixes all "natural numbers" of $\mathcal{M}$.
- **(c)** There is a model $\mathcal{M}$ of $\mathrm{NFU}^+ + \mathrm{AxCount}$ such that T is the first order theory of ("the largest rank initial segment of the Cantorian part of **V**")$^{\mathcal{M}}$