AN INEVITABLE EXTENSION OF ZFC

Ali Enayat

Kunen Fest, April 3, 2009
Mahlo Cardinals

• A Mahlo cardinal $\kappa$ is a strongly inaccessible cardinal such that the regular cardinals below $\kappa$ form a stationary subset of $\kappa$.

• For an ordinal $\alpha$, the $\alpha$-Mahlo cardinals are defined recursively as follows:

  $\kappa$ is 0-Mahlo if $\kappa$ is strongly inaccessible;

  For $\alpha = \delta + 1$, $\kappa$ is a $\alpha$-Mahlo if

  $\{\gamma < \kappa : \gamma$ is $\delta$-Mahlo\} is stationary in $\kappa$;

  For limit $\alpha$, $\kappa$ is $\alpha$-Mahlo if $\kappa$ is $\delta$-Mahlo for all $\delta < \alpha$. 
Levy and Reflection

• Levy showed that $\Sigma_n$-truth is $\Sigma_n$-definable for $n \geq 1$ within ZF.

• In particular, for each natural number $n$ there is a unary formula with the free variable $\alpha$, denoted “$V_\alpha \prec_n V$”, that expresses:

  for all $\Sigma_n$-formulae $\varphi(v_1, \ldots, v_k)$, and all $a_1, \ldots, a_k$ in $V_\alpha$,

  \[ \varphi(a_1, \ldots, a_k) \leftrightarrow \varphi^{V_\alpha}(a_1, \ldots, a_k). \]

• For a unary formula $C(\alpha)$, possibly with suppressed parameters,

  “$\{\alpha : C(\alpha)\}$ is c.u.b.”

stands for the formula expressing

  “$\{\alpha \in \text{Ord} : C(\alpha)\}$ is c.u.b in $\text{Ord}$”.
• **Reflection Theorem** (Montague 1957; Levy 1960) *For each natural number* $n$, ZF proves that $\{\alpha : V_\alpha \prec_n V\}$ *is c.u.b.*

• **Theorem** (Levy 1960). *For each natural numbers* $n$, *the following statement is provable within* ZF:

$$(\kappa \text{ is (} n+1 \text{-Mahlo}) \rightarrow \exists \alpha < \kappa (\alpha \text{ is } n\text{-Mahlo and } V_\alpha \prec V_\kappa)).$$
The Levy Scheme \( \Lambda \)

- \( \lambda_{m,n}(\kappa) \) is the sentence in set theory asserting that \( \kappa \) is an \( m \)-Mahlo cardinal and \( V_\kappa \prec_n V \).

- \( \Lambda := \{ \exists \kappa \ (\lambda_{n,n}(\kappa) : n \in \omega) \} \).

- \( \Lambda_1 := \{ \forall \alpha \in \text{Ord} \ \exists \kappa > \alpha \ \lambda_{n,n}(\kappa) : n \in \omega \} \).

- \( \Lambda_2 := \{ \forall \alpha \in \text{Ord} \ \exists \kappa > \alpha \ \lambda_{m,n}(\kappa) : m \in \omega, \ n \in \omega \} \).

- \( \Lambda_3 := \{ \psi_{C(\alpha,x),n} : C = C(\alpha, x) \) is a binary formula of set theory\}, where

\[
\psi_{C,n} := \forall x [\{ \alpha \in \text{Ord} : C(\alpha, x) \} \text{ is c.u.b.} \rightarrow \exists \kappa \ C(\kappa, x) \text{ and } \kappa \text{ is } n\text{-Mahlo}] .
\]
Different Faces of $\Lambda$

- **Theorem** (Levy 1960). Over ZF, the theories $\Lambda, \Lambda_1, \Lambda_2,$ and $\Lambda_3$ are pairwise equivalent.

- $\Lambda_0 := \{\exists \kappa \; \kappa \text{ is } n\text{-Mahlo}: n \in \omega\}$.

- **Proposition** (Folklore)
  
  (a) The theories $ZF + \Lambda_0$ and $ZF + \Lambda$ are equiconsistent.

  (b) Moreover, assuming $\text{Con}(ZF + \Lambda_0)$, neither $\Lambda_0$, nor $\Lambda$ is finitely axiomatizable over ZF.
The robustness of $\Lambda$

- **Theorem** If $M \models \text{ZFC} + \Lambda$, and $c \in M$, then $(L(c))^M \models \Lambda$.

- **Theorem** If $M \models \text{ZFC} + \Lambda$ and $\mathbb{P} \in M$ is a partial order, then for every $\mathbb{P}$-generic filter $G$ over $M$, $M[G] \models \Lambda$.

- **Corollary.** Suppose $\text{Con}(\text{ZF} + \Lambda)$. Then for any sentence $\psi$, $\text{Con}(\text{ZF} + \Lambda + \psi)$ if at least one of the following conditions are true:
  
  (a) $\text{ZF} \vdash \text{“} \psi \text{ holds in } L \text{”}$, or
  
  (b) $\text{ZF} \vdash \text{“} \text{ for some poset } \mathbb{P}, 1_\mathbb{P} \models \psi \text{”}$,
Finite Set Theory

• $\text{TC} := \text{"every set has a transitive closure".}$

• $\text{ZF}_{\text{fin}} = \text{ZF}\setminus\{\text{Infinity}\} + \neg\text{Infinity} + \text{TC}.$

• $\text{GBC}_{\text{fin}} = \text{GBC}\setminus\{\text{Infinity}\} + \neg\text{Infinity} + \text{TC}.$

• **Theorem** [Ackernann 1940, Kaye-Wong 2008]

  (a) $\text{ZF}_{\text{fin}}$ is bi-interpretable with PA.

  (b) $\text{GBC}_{\text{fin}}$ is bi-interpretable with $\text{ACA}_0.$
Inevitability of $\land$, Exhibit 1

- Let “Ord is WC” be the statement in class theory asserting that every “Ord-tree” has a branch of length $\text{Ord}$.

- **Theorem** [E 2004]
  
  (a) If $(M, A) \models \text{GBC} + \text{Ord is WC},$ then $M \models \text{ZFC} + \land$.

  (b) Every completion of $\text{ZFC} + \land$ has a countable model that has an expansion to a model of $\text{GBC} + \text{Ord is WC}$.

- **Corollary** $\text{GBC} + \text{Ord is WC}$ is a conservative extension of $\text{ZFC} + \land$.

- **Theorem** (Folklore) $\text{GBC}_{\text{fin}}$ is a conservative extension of $\text{ZF}_{\text{fin}}$. 
Inevitability of \( \land \), Exhibit 2

\begin{itemize}
  \item ZFC(I) is a theory in the language \( \{\in, I(x)\} \), where \( I(x) \) is a unary predicate.
  
  \item The axioms of ZFC(I) are as follows.
    
    (1) ZFC + All instances of replacement (hence separation) in \( \{\in, I(x)\} \);
    
    (2) \( I \) is a cofinal subclass of ordinals;
    
    (3) \( I \) is a class of indiscernibles for \((V, \in)\).
\end{itemize}
• **Theorem** (E 2005). The following are equivalent for a completion $T$ of ZFC:

1. $T$ has a model $M$ that expands to a model $(M, I) \models \text{ZFC}(I)$.

2. $T$ has a model $M$ that expands to $(M, I_n)_{n < \omega}$ satisfying $\text{ZF}((\{I_n : n \in \omega\}) + "I_{n+1} is a set of indiscernibles for (V, I_k)_{k \leq n}"$.

3. $T$ is an extension of $\text{ZFC} + \Lambda$.

• **Remark.** If Replacement ($I$) is weakened to Separation($I$), the resulting system is conservative over ZFC.

• **Theorem** [E 2005] $\text{ZF}_{\text{fin}}(I)$ is a conservative extension of $\text{ZF}_{\text{fin}}$. 
Inevitability of $\Lambda$, Exhibit 3

- **Theorem.** [E 2001] The Continuum Hypothesis is a sufficient, but not a necessary condition for every consistent extension of ZF to have an $\aleph_2$-like model.

- **Theorem** [Kaufmann, E 1984] Every completion of ZFC has a $\theta$-like model for every $\theta \geq \aleph_1$.

- **Theorem.** [E 2001] $\text{Con}(ZF + \text{there is an } \omega\text{-Mahlo cardinal})$ implies consistency of “the only completions of ZFC that have an $\aleph_2$-like model are those containing $\Lambda$”.

- **Theorem** (McDowell-Specker 1961). Every completion of $ZF_{\text{fin}}$ has a $\theta$-like model for every $\theta \geq \aleph_1$. 
Inevitability of $\land$, Exhibit 4

- The theory NFU was introduced by Jensen as a modification of Quine’s elegant formulation $NF$ (New Foundations) of Russell’s theory of types.

- NF is a first order theory whose axioms consist of the *stratifiable* comprehension scheme and the usual extensionality axiom.

- The stratifiable comprehension scheme is the collection of sentences of the form “$\{x : \varphi(x)\}$ exists”, provided there is an integer valued function $f$ whose domain is the set of all variables occurring in $\varphi$, which satisfies the following two requirements: (1) $f(v) + 1 = f(w)$, whenever $(v \in w)$ is a subformula of $\varphi$; (2) $f(v) = f(w)$, whenever $(v = w)$ is a subformula of $\varphi$. 
Jensen’s variant NFU of NF is obtained by modifying the extensionality axiom so as to allow *urelements*.

**Theorem** (Jensen 1968)

(a) \( \text{Con(PA)} \Rightarrow \text{Con(NFU + \neg Infinity)} \).

(b) \( \text{Con(Z)} \Rightarrow \text{Con(NFU + Choice + Infinity)} \).

- \( X \) is *Cantorian* if there is a one-to-one correspondence between \( X \) and \( \{\{v\} : v \in X\} \); \( X \) is *strongly Cantorian* if the map sending \( v \) to \( \{v\} \) (as \( v \) varies in \( X \)) exists;

- \( H := \) “every Cantorian set is strongly Cantorian”

- \( \text{NFUA} := \text{NFU + Infinity + Choice + H} \).

- \( \text{NFUA}_{\text{fin}} := \text{NFUA}\{\text{Infinity}\} + \{\neg\text{Infinity}\} \).
• **Theorem** (Solovay, 1995) $\text{Con}(\text{ZFC}+\Lambda_0) \iff \text{Con}($NFUA$)$.

• **Theorem** (E 2002) The following are equivalent for a theory $T$ in the language $\{\in\}$:

** (a) $T$ is a consistent completion of $\text{ZFC}+\Lambda$.

** (b) There is a model $M$ of $\text{NFUA}$ such that $T = \text{Th}(\text{“Cantorian part of } V\text{”})^M$.

• **Theorem** (Solovay-E, 2002). The analogue of the above theorem holds for $\text{ZF}_{\text{fin}}$ and $\text{NFUA}_{\text{fin}}$, in particular:

\[
\text{Con}(\text{NFUA}_{\text{fin}}) \iff \text{Con}(\text{ZF}_{\text{fin}}).
\]
Inevitability of \( \land \), Exhibit 5

- EST is ZFC \( \{ \text{Power Set, Replacement} \} + \Delta_0 \)-Separation.

- GW is the axiom in the language \( \{ \in, \triangleleft \} \) that is the conjunction of the following 4 axioms:

  (1) \( \triangleleft \) totally orders the universe; (2) Every nonempty set has a \( \triangleleft \)-least element, (3) \( x \in y \rightarrow x \triangleleft y \); (4) \( \forall x \exists y \forall z (z \in y \leftrightarrow z \triangleleft x) \).
Exhibit 5, Continued

• **Theorem** [E 2004]

  (a) For every completion $T$ of $\text{ZFC} + \Lambda$ there is a model $M_0$ of $T + \text{ZF}(\langle \rangle) + \text{GW}$ such that $M_0$ has a proper e.e.e. $M$ such that for some automorphism $f$ of $M$, the fixed point set of $f$ is $M_0$.

  (b) Moreover, if $j$ is an automorphism of $M |\models \text{EST}$ whose fixed point set $M_0$ is a $\langle \rangle$-initial segment of $N$, then $M_0 \models \text{ZFC} + \Lambda$.

• **Theorem**

  (a) (Gaifman) *The analogue of (a) above for $\text{ZF}_{\text{fin}}$.*

  (b) (E 2004) *The analogue of (b) above for $\text{ZF}_{\text{fin}}$ (with $1 - \Delta_0$ instead of EST).*