5.4 Covariance of a Probability Distribution and Its Application in Finance

Section 5.1 defined the expected value, variance, and standard deviation for a single discrete variable. In this section, the covariance between two variables is introduced and applied to portfolio management, a topic of great interest to financial analysts.

Covariance

The **covariance of a probability distribution (σ_{XY})** measures the strength of the relationship between two variables, X and Y. A positive covariance indicates a positive relationship. A negative covariance indicates a negative relationship. If two variables are independent, their covariance will be zero. Equation (5.9) defines the covariance of discrete random variables X and Y.

COVARIANCE

$$\sigma_{XY} = \sum_{i=1}^{N}[x_i - E(X)][y_i - E(Y)]P(x_i, y_i) \tag{5.9}$$

where

X = discrete variable X
x_i = ith value of X
Y = discrete variable Y
y_i = ith value of Y
$P(x_i, y_i)$ = probability of occurrence of the ith value of X and the ith value of Y
i = 1, 2, . . . , N for X and Y

To illustrate the covariance, suppose that you are deciding between two different investments for the coming year. The first investment is a mutual fund that consists of the stocks that comprise the Dow Jones Industrial Average. The second investment is a mutual fund that is expected to perform best when economic conditions are weak. Table 5.4 summarizes your estimate of the returns (per $1,000 investment) under three economic conditions, each with a given probability of occurrence.

TABLE 5.4
Estimated Returns for Each Investment Under Three Economic Conditions

		Investment Return	
$P(x_i, y_i)$	Economic Condition	Dow Jones Fund	Weak-Economy Fund
0.2	Recession	−$300	+$200
0.5	Stable economy	+100	+50
0.3	Expanding economy	+250	−100

The expected value and standard deviation for each investment and the covariance of the two investments are computed as follows:

Let X = the return of the Dow Jones fund and Y = the return of the weak-economy fund

$$E(X) = \mu_X = (-300)(0.2) + (100)(0.5) + (250)(0.3) = \$65$$
$$E(Y) = \mu_Y = (+200)(0.2) + (50)(0.5) + (-100)(0.3) = \$35$$
$$Var(X) = \sigma_X^2 = (-300 - 65)^2(0.2) + (100 - 65)^2(0.5) + (250 - 65)^2(0.3)$$
$$= 37{,}525$$
$$\sigma_X = \$193.71$$

studentTIP
The covariance discussed in this section measures the strength of the linear relationship between the *probability distributions* of two variables, while the *sample* covariance discussed in Chapter 3 measures the strength of the linear relationship between two numerical variables.

$$\begin{aligned}
Var(Y) = \sigma_Y^2 &= (200-35)^2(0.2) + (50-35)^2(0.5) + (-100-35)^2(0.3) \\
&= 11{,}025 \\
\sigma_Y &= \$105.00 \\
\sigma_{XY} &= (-300-65)(200-35)(0.2) + (100-65)(50-35)(0.5) \\
&\quad + (250-65)(-100-35)(0.3) \\
&= -12{,}045 + 262.5 - 7{,}492.5 \\
&= -19{,}275
\end{aligned}$$

Thus, the Dow Jones fund has a higher expected value (i.e., larger expected return) than the weak-economy fund but also has a higher standard deviation (i.e., more risk). The covariance of $-19{,}275$ between the two investment returns indicates a negative relationship in which the return of two investments are varying in the *opposite* direction. Therefore, when the return on one investment is high, typically, the return on the other investment is low.

Expected Value, Variance, and Standard Deviation of the Sum of Two Variables

Equations (5.1) through (5.3) define the expected value, variance, and standard deviation of a probability distribution, and Equation (5.9) defines the covariance between two variables, X and Y. The **expected value of the sum of two variables** is equal to the sum of the expected values. The **variance of the sum of two variables** is equal to the sum of the variances plus twice the covariance. The **standard deviation of the sum of two variables** is the square root of the variance of the sum of two variables.

EXPECTED VALUE OF THE SUM OF TWO VARIABLES

$$E(X+Y) = E(X) + E(Y) \tag{5.10}$$

VARIANCE OF THE SUM OF TWO VARIABLES

$$Var(X+Y) = \sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} \tag{5.11}$$

STANDARD DEVIATION OF THE SUM OF TWO VARIABLES

$$\sigma_{X+Y} = \sqrt{\sigma_{X+Y}^2} \tag{5.12}$$

To illustrate the expected value, variance, and standard deviation of the sum of two variables, consider the two investments previously discussed. If X = return of the Dow Jones fund and Y = return of the weak-economy fund, using Equations (5.10), (5.11), and (5.12),

$$\begin{aligned}
E(X+Y) &= E(X) + E(Y) = 65 + 35 = \$100 \\
\sigma_{X+Y}^2 &= \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} \\
&= 37{,}525 + 11{,}025 + (2)(-19{,}275) \\
&= 10{,}000 \\
\sigma_{X+Y} &= \$100
\end{aligned}$$

The expected value of the sum of the return of the Dow Jones fund and the return of the weak-economy fund is \$100, with a standard deviation of \$100. The standard deviation of the sum of the two investments is less than the standard deviation of either single investment because there is a large negative covariance between the investments.

Portfolio Expected Return and Portfolio Risk

The covariance and the expected value and standard deviation of the sum of two random variables can be applied to analyzing **portfolios**, or groupings of assets made for investment purposes. Investors combine assets into portfolios to reduce their risk (see references 1 and 2). Often, the objective is to try to maximize the return while making the risk as small as possible. For such portfolios, rather than study the sum of two random variables, the investor weights each investment by the proportion of assets assigned to that investment. Equations (5.8) and (5.9) define the **portfolio expected return** and **portfolio risk**.

PORTFOLIO EXPECTED RETURN

The portfolio expected return for a two-asset investment is equal to the weight assigned to asset *X* multiplied by the expected return of asset *X* plus the weight assigned to asset *Y* multiplied by the expected return of asset *Y*.

$$E(P) = wE(X) + (1 - w)E(Y) \tag{5.13}$$

where

$E(P)$ = portfolio expected return
w = portion of the portfolio value assigned to asset X
$(1 - w)$ = portion of the portfolio value assigned to asset Y
$E(X)$ = expected return of asset X
$E(Y)$ = expected return of asset Y

PORTFOLIO RISK

The portfolio risk for a two-asset investment is equal to the square root of the sum of these three products: w^2 multiplied by the variance of X, $(1 - w)^2$ multiplied by the variance of Y, and 2 multiplied by w multiplied by $(1 - w)$ multiplied by the covariance.

$$\sigma_p = \sqrt{w^2\sigma_X^2 + (1 - w)^2\sigma_Y^2 + 2w(1 - w)\sigma_{XY}} \tag{5.14}$$

In the previous section, you evaluated the expected return and risk of two different investments, a Dow Jones fund and a weak-economy fund. You also computed the covariance of the two investments. Now, suppose that you want to form a portfolio of these two investments that consists of an equal investment in each of these two funds. To compute the portfolio expected return and the portfolio risk, using Equations (5.13) and (5.14), with $w = 0.50$, $E(X) = \$65$, $E(Y) = \$35$, $\sigma_X^2 = 37{,}525$, $\sigma_Y^2 = 11{,}025$, and $\sigma_{XY} = -19{,}275$,

$$E(P) = (0.5)(65) + (1 - 0.5)(35) = \$50$$
$$\sigma_p = \sqrt{(0.5)^2(37{,}525) + (1 - 0.5)^2(11{,}025) + 2(0.5)(1 - 0.5)(-19{,}275)}$$
$$= \sqrt{2{,}500} = \$50$$

Thus, the portfolio has an expected return of \$50 for each \$1,000 invested (a return of 5%) and a portfolio risk of \$50. The portfolio risk here is smaller than the standard deviation of either investment because there is a large negative covariance between the two investments. The fact that each investment performs best under different circumstances reduces the overall risk of the portfolio.

Collapses in the financial marketplace that have occurred in the recent past have caused some investors to consider the effect of outcomes that have only a small chance of occurring but that could produce extremely negative results. (Some, including the author of reference 6, have labeled these outcomes "black swans.") Example 5.1 considers such an outcome by examining the expected return, the standard deviation of the return, and the covariance of two investment strategies—one that invests in a fund that does well when there is an extreme recession and the other that invests in a fund that does well under positive economic conditions.

EXAMPLE 5.6

Computing the Expected Return, the Standard Deviation of the Return, and the Covariance of Two Investment Strategies

You plan to invest $1,000 in one of two funds. Table 5.5 shows the annual return (per $1,000) of each of these investments under different economic conditions, along with the probability that each of these economic conditions will occur.

TABLE 5.5
Estimated Returns of Two Funds

Probability	Economic Condition	Black Swan Fund	Good Times Fund
0.01	Extreme recession	400	−200
0.09	Recession	−30	−100
0.15	Stagnation	30	50
0.35	Slow growth	50	90
0.30	Moderate growth	100	250
0.10	High growth	100	225

For the Black Swan fund and the Good Times fund, compute the expected return and standard deviation of the return for each fund, and the covariance between the two funds. Would you invest in the Black Swan fund or the Good Times fund? Explain.

SOLUTION Let X = Black Swan fund and Y = Good Times fund.

$$E(X) = \mu_X = (400)(0.01) + (-30)(0.09) + (30)(0.15) + (50)(0.35) + (100)(0.30) + (100)(0.10) = \$63.30$$

$$E(Y) = \mu_Y = (-200)(0.01) + (-100)(0.09) + (50)(0.15) + (90)(0.35) \quad (250)(0.30) + (225)(0.10) = \$125.50$$

$$Var(X) = \sigma_X^2 = (400 - 63.30)^2(0.01) + (-30 - 63.30)^2(0.09) + (30 - 63.30)^2(0.15) \quad (50 - 63.30)^2(0.35) + (100 - 63.30)^2(0.3) + (100 - 63.30)^2(0.1) = 2{,}684.11$$

$$\sigma_X = \$51.81$$

$$Var(Y) = \sigma_Y^2 = (-200 - 125.50)^2(0.01) + (-100 - 125.50)^2(0.09) + (50 - 125.50)^2(0.15) + (90 - 125.50)^2(0.35) + (250 - 125.50)^2(0.3) + (225 - 125.50)^2(0.1) = 12{,}572.25$$

$$\sigma_Y = \$112.13$$

$$\sigma_{XY} = (400 - 63.30)(-200 - 125.50)(0.01) + (-30 - 63.30)(-100 - 125.50)(0.09) + (30 - 63.30)(50 - 125.50)(0.15) + (50 - 63.30)(90 - 125.50)(0.35) + (100 - 63.30)(250 - 125.50)(0.3) + (100 - 63.30)(225 - 125.50)(0.1)$$

$$\sigma_{xy} = \$3{,}075.85$$

(continued)

Thus, the Good Times fund not only has a much higher expected value (i.e., larger expected return) than the Black Swan fund (\$125.50 as compared to \$63.30 per \$1,000) but also has a much higher standard deviation (\$112.13 vs. \$51.81). Deciding which fund to invest in is a matter of how much risk you are willing to tolerate. Although the Good Times fund has a much higher expected return, many people would be reluctant to invest in a fund where there is a chance of a substantial loss.

The covariance of \$3,075.85 between the two investments indicates a positive relationship in which the two investments are varying in the *same* direction. Therefore, when the return on one investment is high, typically, the return on the other is also high. However, from Table 5.5, you can see that the magnitude of the return varies, depending on the economic condition that actually occurs. Therefore, you might decide to include both funds in your portfolio. The percentage allocated to each fund would be based on your tolerance of risk balanced by your desire for maximum return (see Problem 5.53).

PROBLEMS FOR SECTION 5.4

LEARNING THE BASICS

5.45 Given the following probability distributions for variables X and Y:

$P(x, y)$	X	Y
0.4	100	200
0.6	200	100

Compute

a. $E(X)$ and $E(Y)$.
b. σ_X and σ_Y.
c. σ_{XY}.
d. $E(X + Y)$.

5.46 Given the following probability distributions for variables X and Y:

$P(x, y)$	X	Y
0.2	−100	50
0.4	50	30
0.3	200	20
0.1	300	20

Compute

a. $E(X)$ and $E(Y)$.
b. σ_X and σ_Y.
c. σ_{XY}.
d. $E(X + Y)$.

5.47 Two investments, X and Y, have the following characteristics:

$$E(X) = \$50, E(Y) = \$100, \sigma_X^2 = 9{,}000,$$
$$\sigma_Y^2 = 15{,}000, \text{ and } \sigma_{XY} = 7{,}500.$$

If the weight of portfolio assets assigned to investment X is 0.4, compute the

a. portfolio expected return.
b. portfolio risk.

APPLYING THE CONCEPTS

5.48 The process of being served at a bank consists of two independent parts—the time waiting in line and the time it takes to be served by the teller. Suppose that the time waiting in line has an expected value of 4 minutes, with a standard deviation of 1.2 minutes, and the time it takes to be served by the teller has an expected value of 5.5 minutes, with a standard deviation of 1.5 minutes. Compute the

a. expected value of the total time it takes to be served at the bank.
b. standard deviation of the total time it takes to be served at the bank.

5.49 In the portfolio example in this section (see page 192), half the portfolio assets are invested in the Dow Jones fund and half in a weak-economy fund. Recalculate the portfolio expected return and the portfolio risk if

a. 30% of the portfolio assets are invested in the Dow Jones fund and 70% in a weak-economy fund.
b. 70% of the portfolio assets are invested in the Dow Jones fund and 30% in a weak-economy fund.
c. Which of the three investment strategies (30%, 50%, or 70% in the Dow Jones fund) would you recommend? Why?

5.50 You are trying to develop a strategy for investing in two different stocks. The anticipated annual return for a \$1,000 investment in each stock under four different economic conditions has the following probability distribution:

		Returns	
Probability	**Economic Condition**	**Stock X**	**Stock Y**
0.1	Recession	−100	50
0.3	Slow growth	0	150
0.3	Moderate growth	80	−20
0.3	Fast growth	150	−100

Compute the
a. expected return for stock *X* and for stock *Y*.
b. standard deviation for stock *X* and for stock *Y*.
c. covariance of stock *X* and stock *Y*.
d. Would you invest in stock *X* or stock *Y*? Explain.

5.51 Suppose that in Problem 5.50 you wanted to create a portfolio that consists of stock *X* and stock *Y*. Compute the portfolio expected return and portfolio risk for each of the following percentages invested in stock *X*:
a. 30%
b. 50%
c. 70%
d. On the basis of the results of (a) through (c), which portfolio would you recommend? Explain.

5.52 You are trying to develop a strategy for investing in two different stocks. The anticipated annual return for a $1,000 investment in each stock under four different economic conditions has the following probability distribution:

		Returns	
Probability	**Economic Condition**	**Stock *X***	**Stock *Y***
0.1	Recession	−50	−100
0.3	Slow growth	20	50
0.4	Moderate growth	100	130
0.2	Fast growth	150	200

Compute the
a. expected return for stock *X* and for stock *Y*.
b. standard deviation for stock *X* and for stock *Y*.
c. covariance of stock *X* and stock *Y*.
d. Would you invest in stock *X* or stock *Y*? Explain.

5.53 Suppose that in Example 5.1, you wanted to create a portfolio that consists of the Black Swan fund and the Good Times fund. Compute the portfolio expected return and portfolio risk for each of the following percentages invested in the Black Swan fund:
a. 30%
b. 50%
c. 70%
d. On the basis of the results of (a) through (c), which portfolio would you recommend? Explain.

5.54 You plan to invest $1,000 in a corporate bond fund or in a common stock fund. The following table presents the annual return (per $1,000) of each of these investments under various economic conditions and the probability that each of those economic conditions will occur. Compute the

Probability	Economic Condition	Corporate Bond Fund	Common Stock Fund
0.01	Extreme recession	−200	−999
0.09	Recession	−70	−300
0.15	Stagnation	30	−100
0.35	Slow growth	80	100
0.30	Moderate growth	100	150
0.10	High growth	120	350

a. expected return for the corporate bond fund and for the common stock fund.
b. standard deviation for the corporate bond fund and for the common stock fund.
c. covariance of the corporate bond fund and the common stock fund.
d. Would you invest in the corporate bond fund or the common stock fund? Explain.
e. If you chose to invest in the common stock fund in (d), what do you think about the possibility of losing $999 of every $1,000 invested if there is an extreme recession?

5.55 Suppose that in Problem 5.54 you wanted to create a portfolio that consists of the corporate bond fund and the common stock fund. Compute the portfolio expected return and portfolio risk for each of the following situations:
a. $300 in the corporate bond fund and $700 in the common stock fund.
b. $500 in each fund.
c. $700 in the corporate bond fund and $300 in the common stock fund.
d. On the basis of the results of (a) through (c), which portfolio would you recommend? Explain.

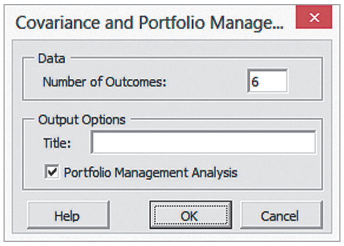
EG5.4 COVARIANCE OF A PROBABILITY DISTRIBUTION AND ITS APPLICATION IN FINANCE

Key Technique Use the **SQRT** and **SUMPRODUCT** functions (see Appendix Section F) to help compute the portfolio analysis statistics.

Example Perform the portfolio analysis for the Section 5.2 investment example.

PHStat Use **Covariance and Portfolio Analysis.**
For the example, select **PHStat ➔ Decision-Making ➔ Covariance and Portfolio Analysis**. In the procedure's dialog box (shown below):

1. Enter **6** as the **Number of Outcomes**.
2. Enter a **Title**, check **Portfolio Management Analysis**, and click **OK**.



In the new worksheet (shown below):

1. Enter the probabilities and outcomes in the table that begins in cell B3.
2. Enter **0.5** as the **Weight assigned to X**.

Workbook Use the **COMPUTE worksheet** of the **Portfolio workbook** as a template.

The worksheet (shown below) already contains the data for the example. Overwrite the X and $P(X)$ values and the weight assigned to the X value when you enter data for other problems. If a problem has more or fewer than three outcomes, first select row **5**, right-click, and click **Insert** (or **Delete**) in the shortcut menu to insert (or delete) rows one at a time. If you insert rows, select the cell range **B4:J4** and copy the contents of this range down through the new table rows.

	A	B	C	D
1	Portfolio Expected Return and Risk			
2				
3	Probabilities & Outcomes:	P	X	Y
4		0.2	-300	200
5		0.5	100	50
6		0.3	250	-100
7				
8	Weight Assigned to X	0.5		
9				
10	Statistics			
11	E(X)	65	=SUMPRODUCT(B4:B6, C4:C6)	
12	E(Y)	35	=SUMPRODUCT(B4:B6, D4:D6)	
13	Variance(X)	37525	=SUMPRODUCT(B4:B6, H4:H6)	
14	Standard Deviation(X)	193.7137	=SQRT(B13)	
15	Variance(Y)	11025	=SUMPRODUCT(B4:B6, I4:I6)	
16	Standard Deviation(Y)	105	=SQRT(B15)	
17	Covariance(XY)	-19275	=SUMPRODUCT(B4:B6, J4:J6)	
18	Variance(X+Y)	10000	=B13 + B15 + 2 * B17	
19	Standard Deviation(X+Y)	100	=SQRT(B18)	
20				
21	Portfolio Management			
22	Weight Assigned to X	0.5	=B8	
23	Weight Assigned to Y	0.5	=1 - B22	
24	Portfolio Expected Return	50	=B22 * B11 + B23 * B12	
25	Portfolio Risk	50	=SQRT(B22^2 * B13 + B23^2 * B15 + 2 * B22 * B23 * B17)	

The worksheet also contains a Calculations Area that contains various intermediate calculations. Open the **COMPUTE_FORMULAS worksheet** to examine all the formulas used in this area.